North Sydney Girls High School

2026 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 8)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – **90 marks** (pages 9 – 16)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

MC	Q11	Q12	Q13	Q14	Q15	Q16	Total
/10	/15	/14	/14	/16	/16	/15	/100

Section I

10 marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1−10.

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1 Consider the following statement.
If $f''(a) = 0$*, then the graph of* $f(x)$ *has a point of inflection at* $x = a$.

Which of the following is equivalent to this statement?

A. If $f''(a) \neq 0$, then $f(x)$ does not have a point of inflection at $x = a$.

B. If $f(x)$ has a point of inflection at $x = a$, then $f''(a) = 0$.

C. If $f(x)$ does not have a point of inflection at $x = a$, then $f''(a) \neq 0$.

D. If $f(x)$ does not have a point of inflection at $x = a$, then $f''(a) = 0$.

2 If the point $z = a + ib$, where $a, b > 0$ is represented as a point on the Argand diagram, the point representing $-\overline{z}$ can be obtained by which of the following?

A. reflecting the point representing z in the real axis.

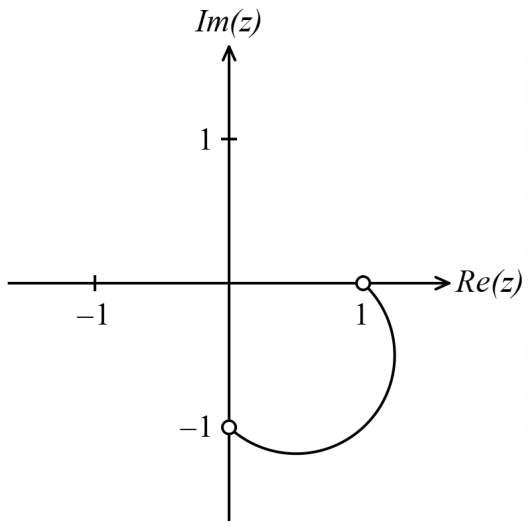
B. reflecting the point representing z in the imaginary axis.

C. rotating the point representing z anti-clockwise about the origin by 90°.

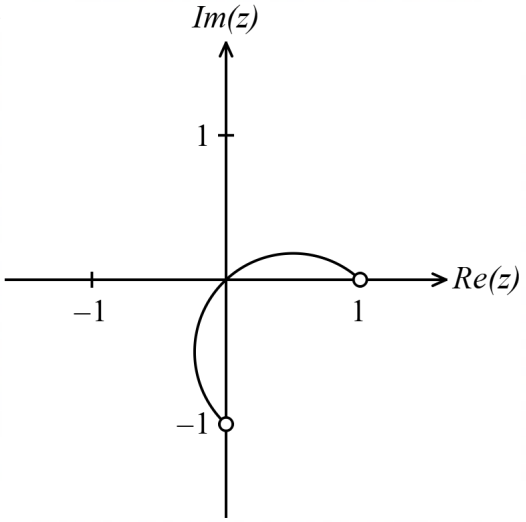
D. rotating the point representing z clockwise about the origin by 90°.

3 Which of the following is the curve in the complex plane defined by $\arg\left(\frac{z-1}{z+i}\right)=\frac{\pi}{2}$?

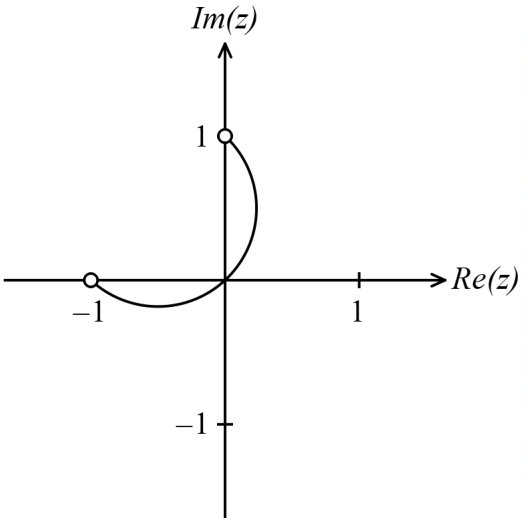
A.



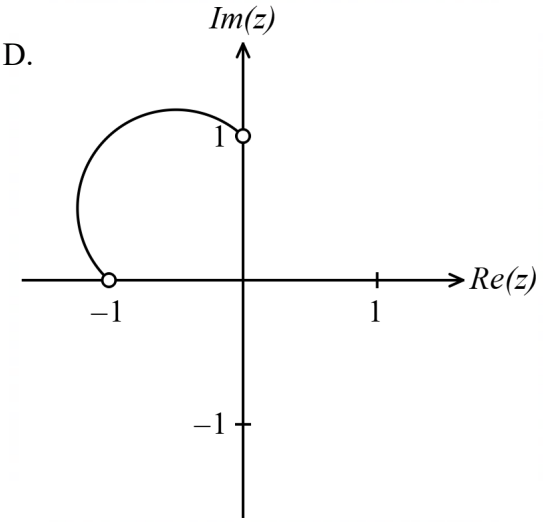
B.



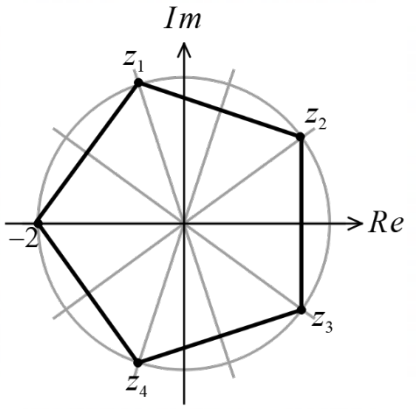
C.



D.



4 Consider the regular polygon below with the complex numbers z_1, z_2, z_3, z_4 and -2 as vertices as shown below.



Which polynomial equation has these complex numbers as roots?

A. $z^4 - 16 = 0$

B. $z^4 + 16 = 0$

C. $z^5 - 32 = 0$

D. $z^5 + 32 = 0$

5 Consider the following statement:

"*Let m be a positive integer. If m is not prime,* $(m-2)$ *is not prime*"

Which of the following values of *m* is a counterexample for this statement?

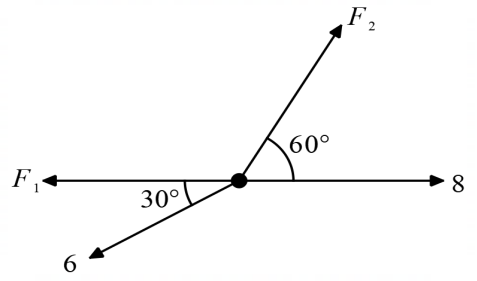
A. 9

B. 12

C. 13

D. 23

6 The diagram below shows a stationary object being acted on by four forces whose magnitudes are in Newtons. The force of magnitude F_1 Newtons acts in the opposite direction to the force of 8 Newtons.



NOT TO SCALE

What is the value of F_1 if the system is in equilibrium?

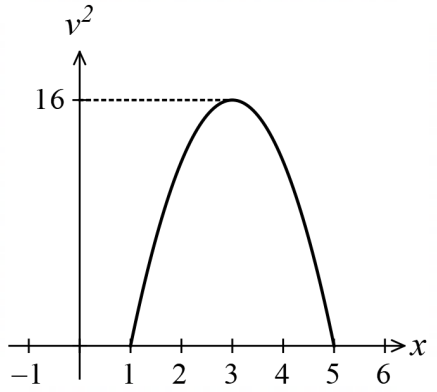
A. $2\sqrt{3}$

B. $8-2\sqrt{3}$

C. 8

D. $8+2\sqrt{3}$

7 A particle is moving in a straight line with displacement x at time t and velocity v. The graph below shows v^2 as a function of x.



Which of the following is a possible equation for the displacement of the particle?

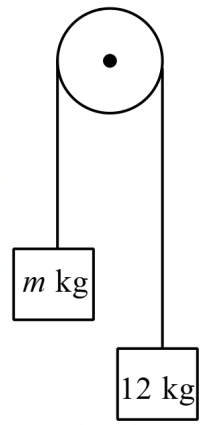
A. $x=3+4\cos(\pi t+\alpha)$

B. $x=3+2\cos(\pi t+\alpha)$

C. $x=3+4\cos(2t+\alpha)$

D. $x=3+2\cos(2t+\alpha)$

8 A light inextensible string passes over a smooth pulley. Attached to the ends of a string are masses of 12 kg and m kg as shown.



The acceleration due to gravity is 10 m/s^2.
The masses are released from rest and the 12 kg mass accelerates downward at 2 m/s^2.

What is the value of m?

A. 8

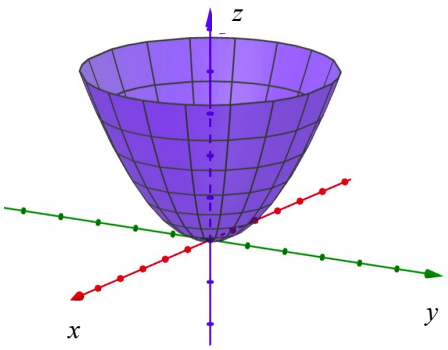
B. 10

C. 12

D. 16

9 The diagram below shows a parabolic bowl formed by rotating a parabola around the z-axis. Its vertex is at the origin of a three-dimensional coordinate system and cross-sections parallel to the xy -plane are circular.

The height of the bowl is 4 units and at the top of the bowl the diameter of the cross-section is 6 units.



Which of the following is a possible set of parametric equations that describe this paraboloid.

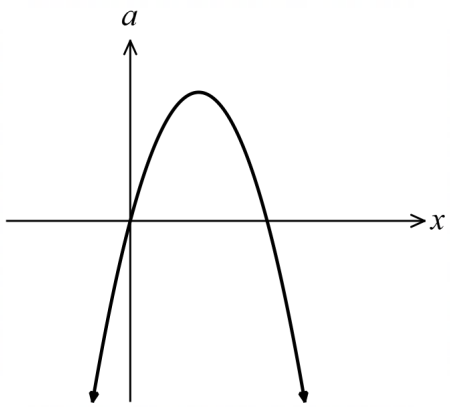
A. $\underset{\sim}{r} = \begin{pmatrix} 3\sqrt{u}\cos v \\ 3\sqrt{u}\sin v \\ u \end{pmatrix}$; $\begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

B. $\underset{\sim}{r} = \begin{pmatrix} \frac{3}{2}u\cos 2v \\ \frac{3}{2}u\sin 2v \\ u^2 \end{pmatrix}$; $\begin{matrix} 0 \le u \le 2 \\ 0 \le v \le \pi \end{matrix}$

C. $\underset{\sim}{r} = \begin{pmatrix} \frac{3}{2}\sqrt{u}\cos v \\ \frac{3}{2}\sqrt{u}\sin v \\ u \end{pmatrix}$; $\begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

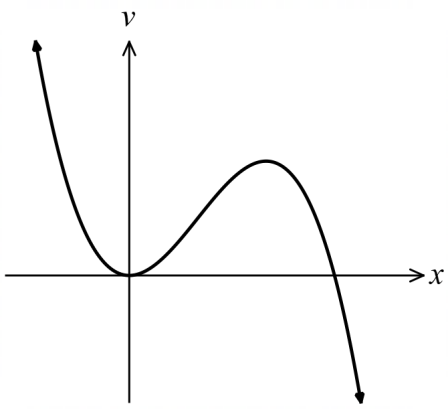
D. $\underset{\sim}{r} = \begin{pmatrix} 3u\cos 2v \\ 3u\sin 2v \\ u^2 \end{pmatrix}$; $\begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

10 The graph below shows the acceleration of a particle as a function of its displacement.

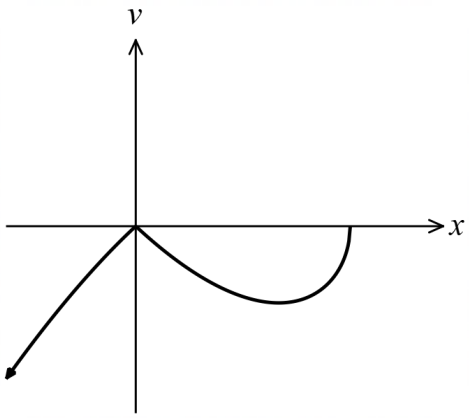


Which of the following best represents a possible graph of its velocity as a function of its displacement?

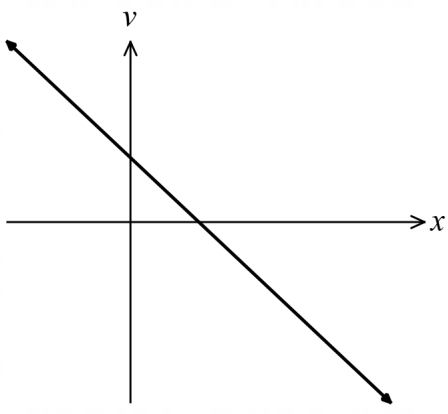
A.



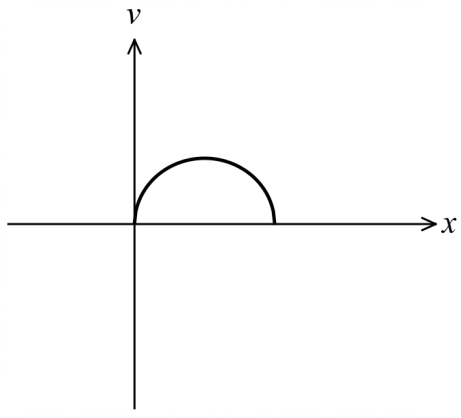
B.



C.



D.



Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet

(a) Let $z = \sqrt{2}e^{\frac{\pi}{4}i}$ and $w = -2 + 2\sqrt{3}i$.

(i) Find the exact value of $\text{Arg}\left(\frac{w}{\overline{z}}\right)$. **2**

(ii) Express $\frac{w}{\overline{z}}$ in $x + iy$ form where x and y are real numbers. **2**
Show all working.

(iii) Hence, deduce that $\tan\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+1}$. **1**

(b) Find $\int \sin^{-1} x \, dx$. **2**

(c) Consider the equation $z^3 + az^2 + bz + 65 = 0$ where $a, b \in \mathbb{R}$. **3**
Given that $z = 2 - 3i$ is a solution to this equation, find the value of ab.

(d) Find $\int_0^{\frac{\pi}{2}} \frac{1}{2 + \cos\theta} \, d\theta$. **3**

(e) Sketch the region in the Argand plane described by the intersection of **2**
$|z - 2i| \leq 2$ and $\frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{3\pi}{4}$.

Question 12 (14 marks) Use a SEPARATE Writing Booklet

(a) Use the method of partial fractions to find $\int \frac{x+1}{6+x-2x^2}dx$. **4**

(b) Use de Moivre's theorem to solve $z^3 = -3\sqrt{2}+3\sqrt{2}i$ giving your answers in modulus-argument form. Show all working. **3**

(c) The lines l_1 and l_2 are given by the vector equations

$\underset{\sim}{r}_1 = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\-2\end{pmatrix}$ and $\underset{\sim}{r}_2 = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\-3\end{pmatrix}$ respectively where $\lambda, \mu \in \mathbb{R}$.

(i) Show that l_1 passes through the point $(3,-2,2)$. **1**

(ii) Determine whether the two lines intersect. **3**

(d) (i) Show that $a^2+b^2 \geq 2ab$ where $a,b \in \mathbb{R}$. **1**

(ii) Hence show that for $a,b,c \in \mathbb{R}^+$, $(a^2+c)(b^2+c) \geq 4abc$. **2**

Question 13 (14 marks) Use a SEPARATE Writing Booklet

(a) A fly moves on a path given by a three dimensional curve. Its position vector from a fixed origin is given by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \cos t \\ \cos 2t \end{pmatrix}$ where $0 \le t \le 2\pi$.

Distances are measured in metres and time in seconds.

(i) Find a unit vector in the direction of the fly's initial position. **1**

(ii) What is the fly's maximum speed? **2**

(b) The displacement $x(t)$ of a particle moving in simple harmonic motion satisfies the equation $\frac{d^2x}{dt^2} = -4x$. Initially the particle is one metre to the left of the origin moving to the right at a speed of $\sqrt{32}$ m/s.

(i) Find the period of the motion. **1**

(ii) Find the displacement function $x(t)$. **3**

(iii) Find the first time the particle achieves its maximum velocity. Answer correct to two decimal places. **2**

(c) Consider the statement below.
For any integers a and b, if $a+b \ge 15$ *then* $a \ge 8$ *or* $b \ge 8$.

(i) Write down the contrapositive of the statement. **1**

(ii) Hence prove the statement. **2**

(d) Given u and v are complex numbers such that $u\overline{v} + \overline{u}v = 0$, prove that $\frac{u}{v}$ is purely imaginary. **2**

Question 14 (16 marks) Use a SEPARATE Writing Booklet

(a) Prove by Mathematical Induction that for integers $n \geq 0$, **3**

$$\left(2^n n!\right)^2 \leq (2n+1)!$$

(b) (i) Find the vector equation of the sphere with a centre at $(5,4,-7)$ and passing through the point $P(3,1,-1)$. **2**

(ii) Find the value(s) of m for which the line $\underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(m^2\underset{\sim}{i} - 5m\underset{\sim}{j} - 3\underset{\sim}{k}\right)$ is tangent to the sphere, at P. **2**

(c) By differentiating $\dfrac{\ln x}{x}$ or otherwise, prove that if $e < a < b$ then $a^b > b^a$. **3**

Question 14 continues on Page 13

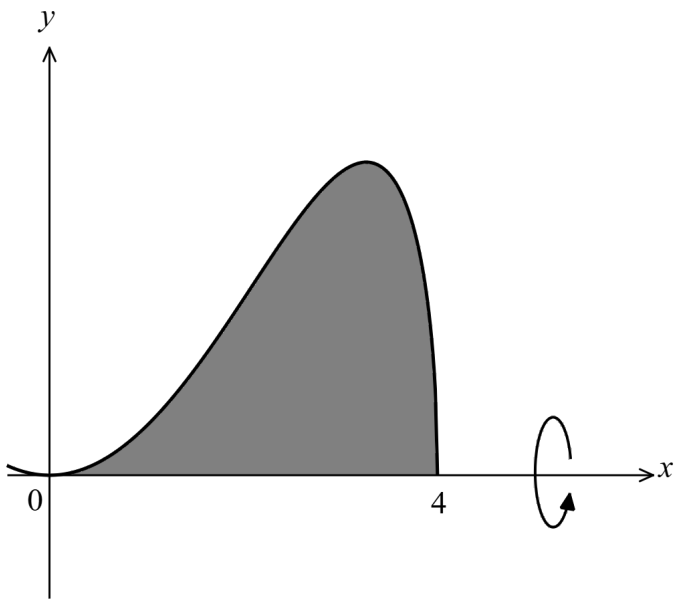
Question 14 (continued)

(d) (i) Given that $I_n = \int_0^4 x^n \sqrt{16 - x^2}\, dx$ for $n \geq 0$, show that **4**

$$(n+2) I_n = 16(n-1) I_{n-2} \text{ for } n \geq 2.$$

(ii) A part of the graph of $f(x) = x^2 \left(16 - x^2\right)^{\frac{1}{4}}$ is shown below. **2**

The region bounded by the curve and the x-axis in the first quadrant is shaded.



Find the exact volume of the solid of revolution formed when the shaded region is rotated about the x-axis.

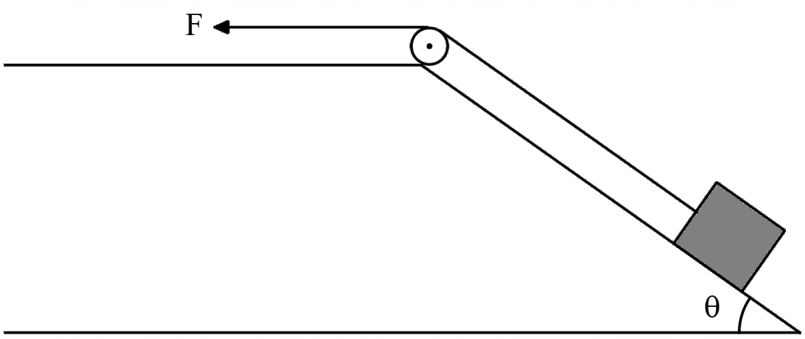
End of Question 14

Question 15 (16 marks) Use a SEPARATE Writing Booklet

(a) Aaron applies a force of F Newtons in the horizontal direction to pull an object of mass 4 kg initially at rest on a ramp, using a light inextensible rope that passes over a smooth pulley as shown.

The ramp is inclined at an angle of θ to the horizontal where $\tan\theta = \frac{3}{4}$.

The coefficient of friction between the object and the ramp is 0.5 and the acceleration due to gravity is 10 m/s^2.



(i) Find the magnitude of the force that Aaron needs to apply so that the object moves with an acceleration of 0.2 m/s^2. **3**

For the following two parts you may assume that the ramp is long enough for the object to move without encountering the pulley.

(ii) How far does the object move in the first 10 seconds? **2**

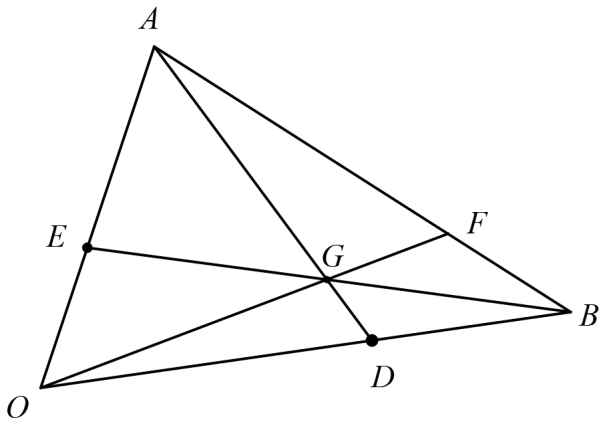
(iii) After 10 seconds, Aaron loses his grip and lets go of the rope. **2**
For how much longer does the object continue to move up the ramp before momentarily coming to rest?

Question 15 continues on the Page 15

(b) A particle moves in a straight line with displacement x and velocity v. **3**
Its acceleration is given by by $\ddot{x} = x - 4$.
Initially the particle is at rest at $x = 1$.

Find an expression for the velocity v in terms of x.

(c) In triangle OAB, the point D divides OB in the ratio $1 : r$ and E divides OA in the ratio $s : 1$. The lines AD and BE intersect at G.



Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(i) Show that $\overrightarrow{OG} = \dfrac{rs}{1+r+rs}\underset{\sim}{a} + \dfrac{1}{1+r+rs}\underset{\sim}{b}$. **4**

(ii) The line through O and G meets AB at F. **2**
Given that F divides AB in the ratio $t : 1$, find an expression for t in terms of r and s.

End of Question 15

Question 16 (15 marks) Use a SEPARATE Writing Booklet

(a) Suppose $f(x)$ and $g(x)$ are continuous functions over $[0,a]$ such that $f(a-x)=f(x)$ and $g(x)+g(a-x)=k$, where k is a constant.

(i) Use a suitable substitution to show that $\int_0^a f(x)g(x)dx=\frac{k}{2}\int_0^a f(x)dx$. **2**

(ii) Use the result in (i) to find $\int_0^2 |x-1|\arccos(x-1)dx$. **3**

(b) Let $\omega=\cos\theta+i\sin\theta$, where $\theta\in[0,2\pi]$.

(i) Show that $\omega^k+\omega^{-k}=2\cos k\theta$, for any integer k. **1**

Let $S=\omega^{-2n}-\omega^{-2n+1}+\omega^{-2n+2}-\ldots-\omega^{-1}+1-\omega+\omega^2-\ldots-\omega^{2n-1}+\omega^{2n}$

(ii) By summing the series show that **3**

$$S=\frac{\omega^{-2n}+\omega^{2n}+\omega^{-(2n+1)}+\omega^{(2n+1)}}{(1+\omega)(1+\bar{\omega})}.$$

(iii) Deduce that

$$1-2(\cos\theta-\cos 2\theta+\cos 3\theta-\ldots-\cos 2n\theta)=\frac{\cos 2n\theta+\cos(2n+1)\theta}{1+\cos\theta}.$$ **2**

(iv) Use the result in (iii) to deduce the value of: **1**

$$\cos\frac{\pi}{13}-\cos\frac{2\pi}{13}+\cos\frac{3\pi}{13}-\ldots+\cos\frac{25\pi}{13}.$$

(c) Prove that there are no non-zero integers a, b, c such that $a^2+b^2+c^2=4abc$. **3**

End of paper

North Sydney Girls High School

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NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

MC	Q11	Q12	Q13	Q14	Q15	Q16	Total
/10	/15	/14	/14	/16	/16	/15	/100

Section I

10 marks
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Use the multiple-choice answer sheet for Questions 1−10.

---

1 Consider the following statement.

If $f''(a)=0$, *then the graph of* $f(x)$ *has a point of inflection at* $x=a$.

Which of the following is equivalent to this statement?

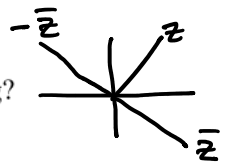
A. If $f''(a)\neq 0$, then $f(x)$ does not have a point of inflection at $x=a$.

B. If $f(x)$ has a point of inflection at $x=a$, then $f''(a)=0$.

C. If $f(x)$ does not have a point of inflection at $x=a$, then $f''(a)\neq 0$. contrapositive

D. If $f(x)$ does not have a point of inflection at $x=a$, then $f''(a)=0$.

2 If the point $z=a+ib$, where $a,b>0$ is represented as a point on the Argand diagram, the point representing $-\bar{z}$ can be obtained by which of the following?



A. reflecting the point representing z in the real axis.

B. reflecting the point representing z in the imaginary axis.

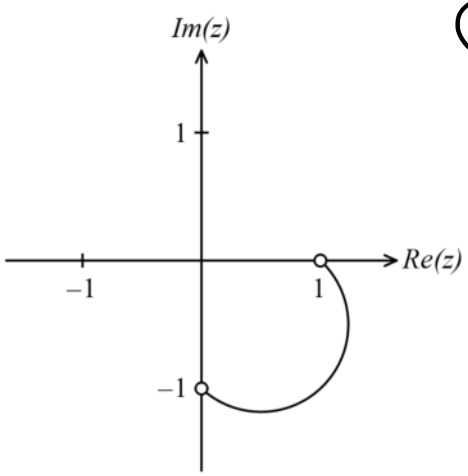
C. rotating the point representing z anti-clockwise about the origin by 90°.

D. rotating the point representing z clockwise about the origin by 90°.

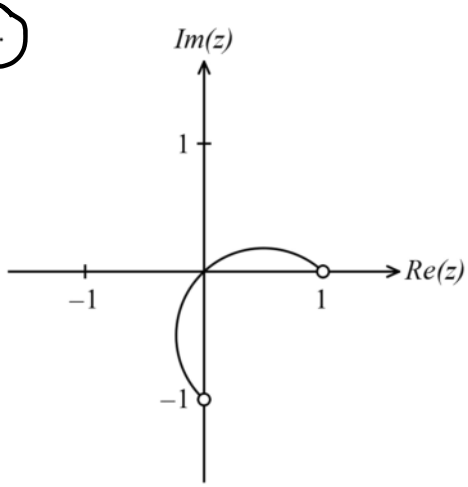
$z=a+ib$
$\bar{z}=a-ib$
$-\bar{z}=-a+ib$

3 Which of the following is the curve in the complex plane defined by $\arg\left(\frac{z-1}{z+i}\right)=\frac{\pi}{2}$?

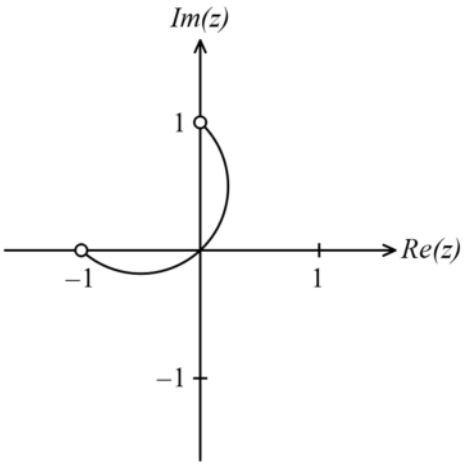
A.



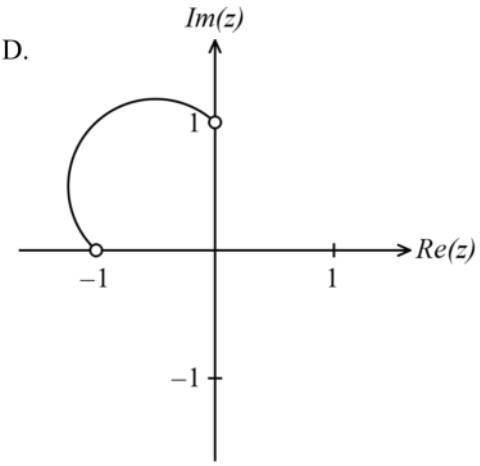
B.



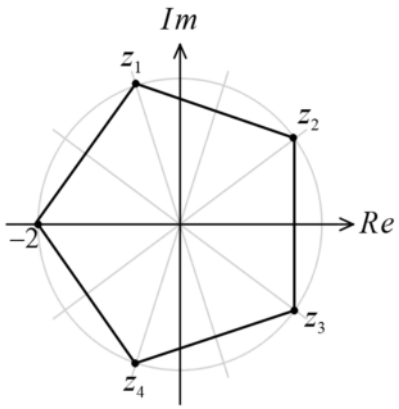
C.



D.



4 Consider the regular polygon below with the complex numbers z_1, z_2, z_3, z_4 and -2 as vertices as shown below.



Which polynomial equation has these complex numbers as roots?

$(-2)^5 = -32$

root of $z^5 = -32$

or $z^5 + 32 = 0$

A. $z^4 - 16 = 0$

B. $z^4 + 16 = 0$

C. $z^5 - 32 = 0$

(D.) $z^5 + 32 = 0$

5 Consider the following statement:

"*Let m be a positive integer. If m is not prime,* $(m-2)$ *is not prime*"

Which of the following values of *m* is a counterexample for this statement?

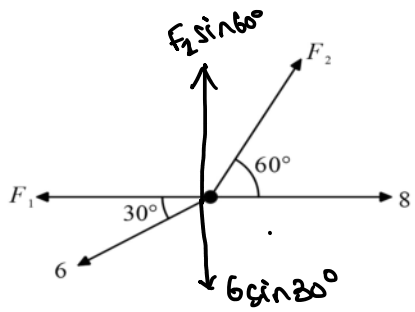
(A.) 9 9 is not prime but $9-2=7$ is prime.

B. 12

C. 13

D. 23

6 The diagram below shows a stationary object being acted on by four forces whose magnitudes are in Newtons. The force of magnitude F_1 Newtons acts in the opposite direction to the force of 8 Newtons.



NOT TO SCALE

$F_1 \sin 60° = 6 \sin 30°$

$F_1 \frac{\sqrt{3}}{2} = 3$

$F_1 =$

What is the value of F_1?

The system is in equilibrium.

Vertically $F_2 \sin 60° = 6 \sin 30°$

$F_2 = \frac{3 \times 2}{\sqrt{3}}$

$= 2\sqrt{3}$

Horizontally $F_1 + 6\cos 30° = 8 + F_2 \cos 60°$

$F_1 = 8 + 2\sqrt{3} \times \frac{1}{2} - 6 \times \frac{\sqrt{3}}{2}$

$= 8 - 2\sqrt{3}$

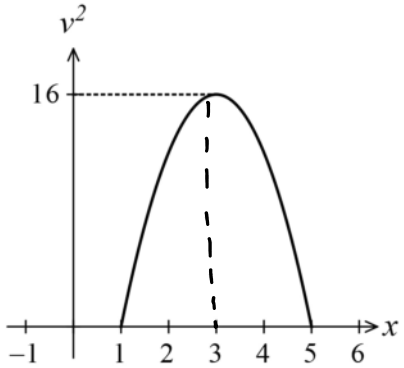
A. $2\sqrt{3}$

B. $8 - 2\sqrt{3}$

C. 8

D. $8 + 2\sqrt{3}$

7 A particle is moving in a straight line with displacement x at time t and velocity v. The graph below shows v^2 as a function of x.



Which of the following is a possible equation for the displacement of the particle?

Amplitude $= \frac{5-1}{2} = 2$

Max velocity $= An$

$A^2n^2 = 16$

$An = 4$

$n = 2$

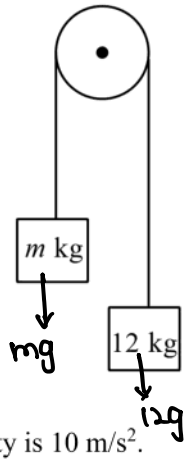
A. $x = 3 + 4\cos(\pi t + \alpha)$

B. $x = 3 + 2\cos(\pi t + \alpha)$

C. $x = 3 + 4\cos(2t + \alpha)$

D. $x = 3 + 2\cos(2t + \alpha)$

8 A light inextensible string passes over a smooth pulley. Attached to the ends of a string are masses of 12 kg and m kg as shown.



The acceleration due to gravity is 10 m/s^2.
The masses are released from rest and the 12 kg mass accelerates downward at 2 m/s^2.

What is the value of m?

$12g - mg = (12+m) \times a \quad a = 2$

$120 - 10m = 24 + 2m$

$12m = 96$

$m = 8$

A. 8

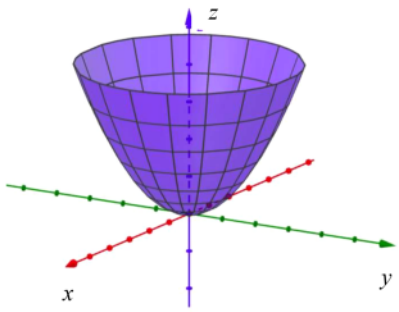
B. 10

C. 12

D. 16

9 The diagram below shows a parabolic bowl formed by rotating a parabola around the z-axis. Its vertex is at the origin of a three-dimensional coordinate system and cross-sections parallel to the xy -plane are circular.

The height of the bowl is 4 units and at the top of the bowl the diameter of the cross-section is 6 units.



Which of the following is a possible set of parametric equations that describe this paraboloid.

A. $\underset{\sim}{r} = \begin{pmatrix} 3\sqrt{u}\cos v \\ 3\sqrt{u}\sin v \\ u \end{pmatrix}; \quad \begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

B. $\underset{\sim}{r} = \begin{pmatrix} \frac{3}{2}u\cos 2v \\ \frac{3}{2}u\sin 2v \\ u^2 \end{pmatrix}; \quad \begin{matrix} 0 \le u \le 2 \\ 0 \le v \le \pi \end{matrix}$

$x^2+y^2 = \frac{9}{4}u^2(\cos^2 2v+\sin^2 2v)$

when $u=2$ $\quad z=u^2=4$

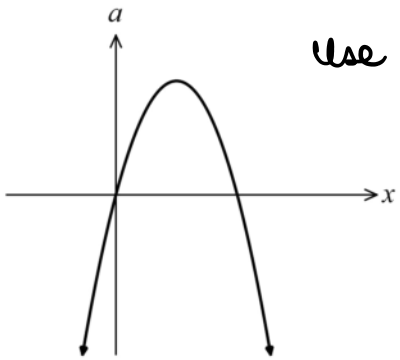
$x^2+y^2=9$ so radius $=3$

diameter $=6$

C. $\underset{\sim}{r} = \begin{pmatrix} \frac{3}{2}\sqrt{u}\cos v \\ \frac{3}{2}\sqrt{u}\sin v \\ u \end{pmatrix}; \quad \begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

D. $\underset{\sim}{r} = \begin{pmatrix} 3u\cos 2v \\ 3u\sin 2v \\ u^2 \end{pmatrix}; \quad \begin{matrix} 0 \le u \le 4 \\ 0 \le v \le \pi \end{matrix}$

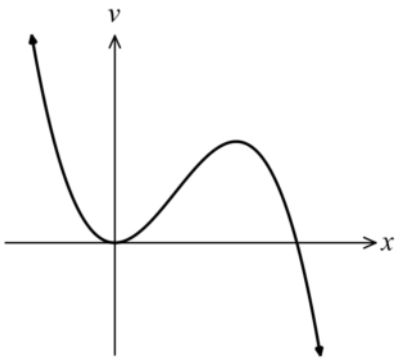
10 The graph below shows the acceleration of a particle as a function of its displacement.



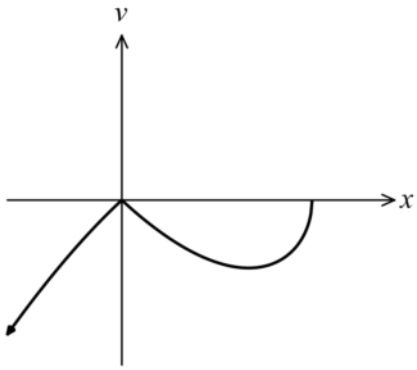
use $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Which of the following best represents a possible graph of its velocity as a function of its displacement?

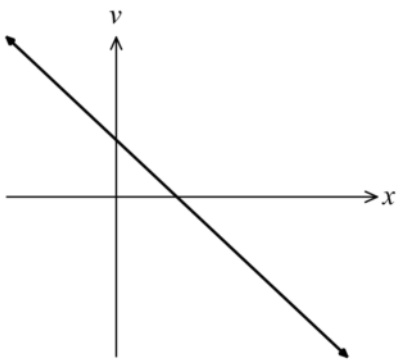
A.



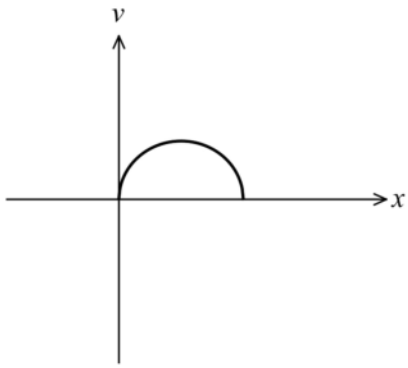
B.



C.



D.



Question 11

(a) $z = \sqrt{2}e^{\frac{\pi}{4}i}$ $\quad w = -2 + 2\sqrt{3}i$

(i) $\text{Arg}\left(\frac{w}{z}\right)$

$\tan(\arg w) = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$



$\arg w = \frac{2\pi}{3}$

$\arg \bar{z} = -\arg z$

$= -\frac{\pi}{4}$

$\arg\left(\frac{w}{z}\right) = \arg w - \arg \bar{z}$

$= \frac{2\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= \frac{2\pi}{3} + \frac{\pi}{4}$

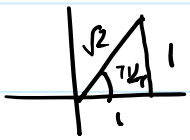
$= \frac{11\pi}{12}$ $\quad \therefore \text{Arg}\left(\frac{w}{z}\right) = \frac{11\pi}{12}$

(ii) $\frac{w}{\bar{z}} = \frac{wz}{z\bar{z}}$

$= \frac{wz}{|z|^2}$

$z = \sqrt{2}e^{\frac{\pi}{4}i}$

$= 1 + i$



$|z| = \sqrt{2}$

$= \frac{(-2+2\sqrt{3}i)(1+i)}{2}$

$= \frac{-2-2i+2\sqrt{3}i-2\sqrt{3}}{2}$

$= -(1+\sqrt{3}) + i(\sqrt{3}-1)$

Question 11

a) (iii) $\tan\frac{11\pi}{12} = \frac{\sqrt{3}-1}{-(\sqrt{3}+1)}$ as $\text{Arg}\left(\frac{z}{\bar{\omega}}\right) = \frac{11\pi}{12}$

$\tan\frac{\pi}{12} = -\tan\frac{11\pi}{12}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$ as required.

b) $\int \sin^{-1}x\,dx = \int \sin^{-1}x\,d(x)$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$ by parts

$= x\sin^{-1}x + \frac{1}{2}\int(-2x)(1-x^2)^{-\frac{1}{2}}dx$

$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

c) $z^3 + az^2 + bz + 65 = 0$

$2-3i$ is a root. $\therefore$ $2+3i$ is also a root

(real coefficients, conjugate roots)

product of roots: $(2+3i)(2-3i)\alpha = -65$

$13\alpha = -65$

$\alpha = -5$

Question 11

c) sum of roots: $2+3i+2-3i+(-5) = -a$

$a = 1$

sum of roots in pairs:

$(2+3i)(-5) + (2-3i)(-5) + (2+3i)(2-3i) = b$

$-10-15i-10+15i+4+9 = b$

$b = -7$

So $ab = (1)(-7)$

$= -7$

d) $\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos\theta}\, d\theta$

$= \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{2\,dt}{2(1+t^2)+1-t^2}$

$= 2\int_0^1 \frac{dt}{3+t^2}$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3\sqrt{3}}$

Let $t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1}t$

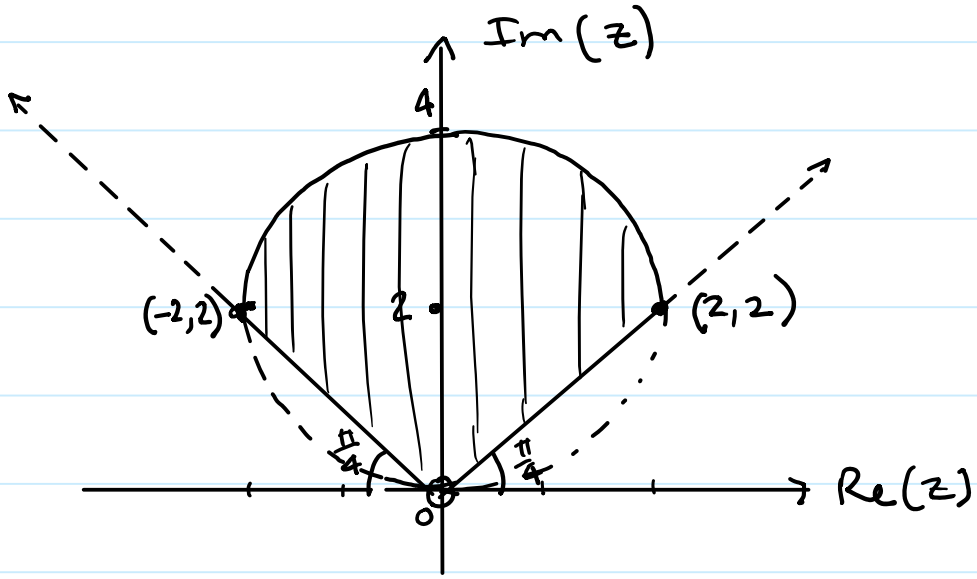
$d\theta = \frac{2\,dt}{1+t^2}$

θ	0	$\frac{\pi}{2}$
t	0	1

$\cos\theta = \frac{1-t^2}{1+t^2}$

Question 11

(e)



Question 12

a) $$\frac{x+1}{6+x-2x^2} = \frac{x+1}{(2-x)(3+2x)} = \frac{A}{2-x} + \frac{B}{3+2x}$$

Equating numerators $\quad x+1 = A(3+2x) + B(2-x)$

Sub $x=2$: $\quad 3 = 7A \quad$ so $A = \frac{3}{7}$

Sub $x=-\frac{3}{2}$: $\quad -\frac{1}{2} = \frac{7}{2}B \quad$ so $B = -\frac{1}{7}$

$$\therefore \frac{x+1}{6+x-2x^2} = \frac{3}{7}\left(\frac{1}{2-x}\right) + \left(\frac{-1}{7}\right)\left(\frac{1}{3+2x}\right)$$

$$\int \frac{x+1}{6+x-2x^2}\,dx = \frac{3}{7}\int \frac{dx}{2-x} - \frac{1}{7}\int \frac{dx}{3+2x}$$

$$= -\frac{3}{7}\ln|2-x| - \frac{1}{14}\ln|3+2x| + C$$

b) $$z^3 = -3\sqrt{2} + 3\sqrt{2}\,i$$

$$= 3\sqrt{2}(-1+i)$$

$$= 6\,\mathrm{cis}\,\frac{3\pi}{4} \qquad \text{Let } z = r\,\mathrm{cis}\,\theta$$

$$(r\,\mathrm{cis}\,\theta)^3 = 6\,\mathrm{cis}\left(\frac{3\pi}{4} + 2k\pi\right) \qquad k \in \mathbb{Z}$$

$$r^3\,\mathrm{cis}\,3\theta = 6\,\mathrm{cis}\left(\frac{3\pi}{4} + 2k\pi\right)$$

$r^3 = 6$ and $3\theta = \frac{3\pi}{4} + 2k\pi$

$r = \sqrt[3]{6}$ $\qquad \theta = \frac{\pi}{4} + \frac{2k\pi}{3} = \frac{(8k+3)\pi}{12}$

$z = \sqrt[3]{6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad k=0$

$= \sqrt[3]{6}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \quad k=1$

$= \sqrt[3]{6}\left(\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right) \quad k=-1$

Question 12

(c) $\underset{\sim}{r}_1 = \begin{pmatrix} 4+\lambda \\ 3+5\lambda \\ -2\lambda \end{pmatrix}$ $\underset{\sim}{r}_2 = \begin{pmatrix} 2+\mu \\ -1-2\mu \\ 3-3\mu \end{pmatrix}$

(i) Testing if $(3,-2,2)$ lies on l_1

x: $3 = 4+\lambda \Rightarrow \lambda = -1$

y: $-2 = 3+5\lambda \Rightarrow 5\lambda = -5$ or $\lambda = -1$

z: $2 = -2\lambda \Rightarrow \lambda = -1$

As the λ values are consistent for:

x, y & z coordinates $(3,-2,2)$ lies on l_1

(ii) Solving simultaneously for point of intersection

$4+\lambda = 2+\mu$ (1)

$3+5\lambda = -1-2\mu$ (2)

$-2\lambda = 3-3\mu$ (3)

From (1) $\mu = \lambda + 2$ Sub into (3)

$-2\lambda = 3-3(\lambda+2)$

$-2\lambda = 3-3\lambda-6$

$\lambda = -3$

Then $\mu = -3+2$

$= -1$

Test in (2) LHS $= 3+5(-3)$ RHS $= -1-2(-1)$

$= -12$ $= 1$

LHS $\neq$ RHS So y-coordinate does not match

$\therefore$ l_1 & l_2 do not intersect.

Question 12

d) (i) Consider $(a-b)^2 \geq 0$ (perfect square)

$$a^2+b^2-2ab \geq 0$$

$$a^2+b^2 \geq 2ab$$

(ii) $a^2+c \geq 2a\sqrt{c}$ using (i)

$b^2+c \geq 2b\sqrt{c}$ using (i)

multiplying both inequalities (LHS, RHS positive)

$$(a^2+c)(b^2+c) \geq (2a\sqrt{c})(2b\sqrt{c})$$

$$= 4abc$$

$\therefore (a^2+c)(b^2+c) \geq 4abc$ as required.

Method 2

$$\text{LHS} = (a^2+c)(b^2+c) = a^2b^2+c^2+c(a^2+b^2)$$

$$\geq 2\sqrt{a^2b^2c^2} + c(2ab) \quad \text{using (i)}$$

$$= 2abc + 2abc$$

$$= 4abc$$

$$= \text{RHS}$$

$\therefore (a^2+c)(b^2+c) \geq 4abc$

Question 13

a) $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \cos t \\ \cos 2t \end{pmatrix}$

(i) Initial position vector $\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

A unit vector in this direction is $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(ii) $\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} \cos t \\ -\sin t \\ -2\sin 2t \end{pmatrix}$

speed $= \left|\dot{\underset{\sim}{r}}(t)\right|$

$= \sqrt{\cos^2 t + \sin^2 t + 4\sin^2 2t}$

$= \sqrt{1 + 4\sin^2 2t}$

max speed $= \sqrt{1+4}$ $\quad 0 \le \sin^2 2t \le 1$

$= \sqrt{5}$ m/s

b)(i) $\ddot{x} = -4x$

$n^2 = 4$

$n = 2$

$\therefore$ period $= \frac{2\pi}{n} = \frac{2\pi}{2}$

$= \pi$

Question 13

b) (ii) Let $x(t) = A\sin(2t-\alpha)$

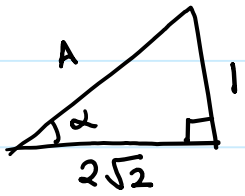
$\dot{x}(t) = 2A\cos(2t-\alpha)$

$x(0) = -1$ so $A\sin(-\alpha) = -1$

$A\sin\alpha = 1$ ①

$\dot{x}(0) = \sqrt{32}$ so $2A\cos\alpha = \sqrt{32}$

$A\cos\alpha = 2\sqrt{2}$ ②



① ÷ ② $\tan\alpha = \frac{1}{2\sqrt{2}}$

$A^2 = 1^2 + (2\sqrt{2})^2$ Pythagoras

$A = 3$

so $x(t) = 3\sin\left(2t - \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)\right)$

(iii) Maximum velocity at centre of motion

ie. at the origin

$x(t) = 0 \Rightarrow 3\sin(2t-\alpha) = 0$ $\alpha = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\sin(2t-\alpha) = 0$

$2t - \alpha = 0, \pi, 2\pi, \ldots$

D

$2t = \alpha, \pi+\alpha, 2\pi+\alpha, \ldots$

$t = \frac{\alpha}{2}, \frac{\pi+\alpha}{2}, \frac{2\pi+\alpha}{2}, \ldots$

First time with max velocity at $t = \frac{1}{2}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\doteq 0.1699\ldots$

ie. at $t = 0.17$ seconds (2dp)

Question 13

(c) (i) Given $a, b \in \mathbb{Z}$, if $a+b \geq 15$ then $a \geq 8$ or $b \geq 8$

Contrapositive:

For $a, b \in \mathbb{Z}$, if $a < 8$ and $b < 8$, then $a+b < 15$

(ii) Let $a < 8$ and $b < 8$

so $a \leq 7$ and $b \leq 7$ as $a, b \in \mathbb{Z}$

Adding we get

$$a+b \leq 14$$

or $$a+b < 15$$

So $a < 8$ and $b < 8 \Rightarrow a+b < 15$

By the contrapositive, If $a+b \geq 15$ then $a \geq 8$ or $b \geq 8$

(d) Given $u\bar{v} + \bar{u}v = 0$

$$u\bar{v} = -\bar{u}v$$

Then $$\frac{u}{v} = -\frac{\bar{u}}{\bar{v}}$$

$$= -\overline{\left(\frac{u}{v}\right)}$$

So $$\frac{u}{v} + \overline{\left(\frac{u}{v}\right)} = 0$$

Then $2\,\mathrm{Re}\left(\frac{u}{v}\right) = 0$ as $z + \bar{z} = 2\mathrm{Re}(z)$

$$\mathrm{Re}\left(\frac{u}{v}\right) = 0 \Rightarrow \frac{u}{v} \text{ is purely imaginary}$$

OR

$u\bar{v} + \overline{u\bar{v}} = 0$ so $2\,\mathrm{Re}(u\bar{v}) = 0$

$u\bar{v}$ is imaginary then $\frac{u\bar{v}v}{v} = \frac{u}{v}|v|^2$ is imaginary

As $|v|^2$ is real, then $\frac{u}{v}$ is imaginary

Question 14

(a) Let $S(n): (2^n n!)^2 \leq (2n+1)!$

Test $n=0$

$LHS = (2^0 \cdot 0!)^2 = (1 \times 1)^2 = 1$

$RHS = (2(0)+1)! = 1! = 1$

$LHS \leq RHS$

Assume $S(k)$ ie. $(2^k k!)^2 \leq (2k+1)!$

Prove $S(k+1)$ RTP $(2^{k+1}(k+1)!)^2 \leq (2k+3)!$

$LHS = (2^{k+1}(k+1)!)^2$

$= (2 \cdot 2^k \cdot (k+1) k!)^2$

$= 2^2 (k+1)^2 (2^k k!)^2$

$\leq 2(k+1) \cdot 2(k+1)(2k+1)!$ using assumption

$= (2k+2)(2k+2)(2k+1)!$

$\leq (2k+3)(2k+2)(2k+1)!$

$= (2k+3)!$

$= RHS$ so $LHS \leq RHS$ & $S(k+1)$ is true

So $S(k) \Rightarrow S(k+1)$

Hence $S(n)$ is true by mathematical induction

Question 14

b) (i) centre $= (5, 4, -7)$

radius $= \left| \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix} \right|$

$= \sqrt{(-2)^2 + (-3)^2 + 6^2}$

$= \sqrt{49}$

$= 7$

$\therefore$ Equation of sphere is $\left| \underset{\sim}{r} - \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} \right| = 7$

(ii) radius is perpendicular to tangent at point of contact

$\therefore \quad \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} m^2 \\ -5m \\ -3 \end{pmatrix} = 0$

$-2m^2 + 15m - 18 = 0$

$2m^2 - 15m + 18 = 0$

$(2m - 3)(m - 6) = 0$

$m = \frac{3}{2}, 6$

Question 14

(c) $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{\frac{1}{x} \times x - \ln x \times 1}{x^2} = \frac{1-\ln x}{x^2}$

$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = 0$ when $1 - \ln x = 0$

$\ln x = 1$

$x = e$

For $x > e$ $\quad 1 - \ln x < 0$ $\quad$ so $\frac{d}{dx}\left(\frac{\ln x}{x}\right) < 0$

$\therefore \frac{\ln x}{x}$ is decreasing for $x > e$

So if $e < a < b$ then $\frac{\ln a}{a} > \frac{\ln b}{b}$

Then $b \ln a > a \ln b$

$\ln a^b > \ln b^a$

so $a^b > b^a$

as $\ln(x)$ is an increasing function.

Question 14

(d) (i)

$$I_n = \int_0^4 x^n \sqrt{16-x^2}\, dx$$

$u = x^{n-1}$ $\quad v = x\sqrt{16-x^2}$

$u' = (n-1)x^{n-2}$ $\quad v = -\frac{1}{2}\left(16-x^2\right)^{\frac{3}{2}} \div \frac{3}{2}$

$= -\frac{1}{3}\left(16-x^2\right)^{3/2}$

$$I_n = \left[x^{n-1} \cdot -\tfrac{1}{3}\left(16-x^2\right)^{3/2}\right]_0^4 - \int_0^4 (n-1)x^{n-2} \times -\tfrac{1}{3}\left(16-x^2\right)^{3/2} dx$$

$$= 0 - 0 + \frac{n-1}{3}\int_0^4 x^{n-2}\left(16-x^2\right)\sqrt{16-x^2}\, dx$$

$$= \frac{n-1}{3}\int_0^4 16x^{n-2}\sqrt{16-x^2} - x^n\sqrt{16-x^2}\, dx$$

$$= 16\left(\frac{n-1}{3}\right) I_{n-2} - \frac{n-1}{3} I_n$$

$$I_n + \frac{n-1}{3} I_n = 16\left(\frac{n-1}{3}\right) I_{n-2}$$

$$\frac{(3+n-1)}{3} I_n = 16\left(\frac{n-1}{3}\right) I_{n-2}$$

$$(n+2) I_n = 16(n-1) I_{n-2} \quad \text{as required.}$$

Question 14

(ii) $V = \pi \int_0^4 y^2 \, dx$ $\qquad y = x^2(16-x^2)^{\frac{1}{4}}$

$y^2 = x^4(16-x^2)^{\frac{1}{2}}$ or $x^4\sqrt{16-x^2}$

$= \pi \int_0^4 x^4\sqrt{16-x^2} \, dx$

$= \pi I_4$

Now using (i)

$6 I_4 = 16(3) I_2$

$I_4 = \frac{\cancel{16} \times \cancel{3}}{\cancel{6}\cancel{2}} I_2$

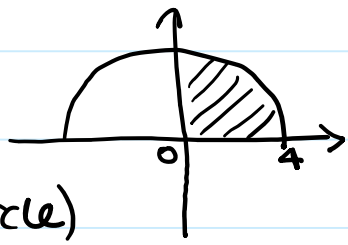
$= 8 I_2$

and $4 I_2 = 16(1) I_0$

So $I_4 = 2 \times 16 I_0$

$= 32 I_0$

$I_0 = \int_0^4 \sqrt{16-x^2} \, dx$



$= \frac{\pi \times 4^2}{4}$ (quarter circle)

$= 4\pi$

So $V = \pi I_4$

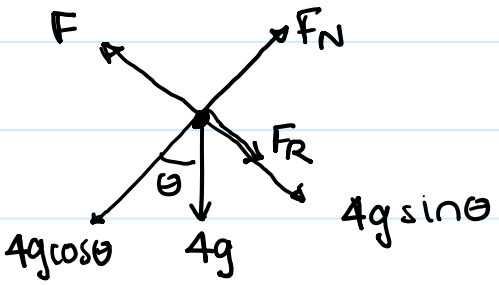
$= \pi(32 I_0)$

$= \pi \times 32 \times 4\pi$

$= 128\pi^2$ units^3

Question 15

a)
(i)

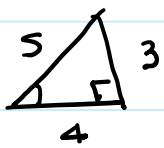


F_N = normal force

F_R = friction force

Perpendicular to plane: $F_N - 4g\cos\theta = 0$

$F_N = 4g\cos\theta$



$\tan\theta = \frac{3}{4}$ $\quad \therefore \cos\theta = \frac{4}{5}$ & $g = 10$

$F_N = 4(10) \times \frac{4}{5}$

$= 32$

So $F_R = 0.5 \times 32$ $\quad (F_R = \mu . F_N)$

$= 16$

Parallel to plane: $F - 4g\sin\theta - F_R = 4a$

$F = 4a + 4g\sin\theta + F_R$

$F_R = 16$, $a = 0.2$, $\sin\theta = \frac{3}{5}$, $g = 10$

So $F = 4(0.2) + 4(10) \times \frac{3}{5} + 16$

$= 40.8$ N is the magnitude of the reqd force.

Question 15

(ii) $\ddot{x} = \frac{dv}{dt} = 0.2$

Integrate wrt t

$$v = 0.2t + C$$

At $t = 0$ $v = 0$ so $C = 0$

$$v = 0.2t$$

so $\frac{dx}{dt} = 0.2t$

$$x = 0.2\frac{t^2}{2} + C_1$$

At $t = 0$ $x = 0$ so $C_1 = 0$

$$x = 0.1t^2$$

At $t = 10$, $x = 0.1 \times 10^2$

$$= 10$$

So the object moves 10 m.

(iii) At $t = 10$, $v = 0.2 \times 10 = 2$

Now there is no external force

So, $4a = -4g\sin\theta - F_R$

$$= -4(10)\left(\frac{3}{5}\right) - 4(10)\left(\frac{4}{5}\right) \times \frac{1}{2}$$

$$a = -10$$

$$v = -10t + C$$

At $t = 0$ $v = 2$ so $C = 2$

$v = 2 - 10t$ so when $v = 0$ $t = \frac{1}{5}$

So the object travels for another 0.2 seconds

Question 15

b) $\ddot{x} = x - 4$

$v\frac{dv}{dx} = x - 4$

$v\,dv = (x-4)\,dx$

$\int_0^v v\,dv = \int_1^x x-4 \cdot dx$

$\left[\frac{v^2}{2}\right]_0^v = \left[\frac{x^2}{2} - 4x\right]_1^x$

$\frac{v^2}{2} - 0 = \left(\frac{x^2}{2} - 4x\right) - \left(\frac{1}{2} - 4\right)$

$\frac{v^2}{2} = \frac{x^2}{2} - 4x + \frac{7}{2}$

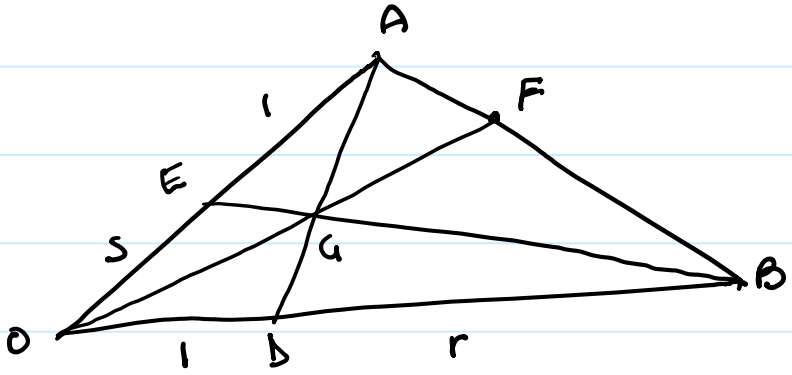
$v^2 = x^2 - 8x + 7$

So $v = \pm\sqrt{x^2 - 8x + 7}$

Initially, it is at rest and $\ddot{x} = 1 - 4 = -3$
So it begins to move in the negative direction
Now $v = 0 \Rightarrow x^2 - 8x + 7 = 0$ ie. $(x-7)(x-1) = 0$
so the particle stops at $x = 1, 7$.
As it is moving left it never reaches $x = 7$.
So it does not turn around and
$v = -\sqrt{x^2 - 8x + 7}$ only.

Question 15

(c)



$\vec{OD} = \frac{1}{1+r}\vec{OB}$

$\vec{OE} = \frac{s}{1+s}\vec{OA}$

(i) $\vec{OG} = \vec{OA} + \vec{AG}$

$= \underset{\sim}{a} + \lambda\vec{AD}$

$= \underset{\sim}{a} + \lambda(\vec{OD} - \vec{OA})$

$= \underset{\sim}{a} + \lambda\left(\frac{1}{1+r}\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= (1-\lambda)\underset{\sim}{a} + \frac{\lambda}{1+r}\underset{\sim}{b}$

Also $\vec{OG} = \vec{OB} + \vec{BG}$

$= \underset{\sim}{b} + \mu\vec{BE}$

$= \underset{\sim}{b} + \mu(\vec{OE} - \vec{OB})$

$= \underset{\sim}{b} + \mu\left(\frac{s}{1+s}\underset{\sim}{a} - \underset{\sim}{b}\right)$

$= \frac{\mu s}{1+s}\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}$

Equating the components

$1-\lambda = \frac{\mu s}{1+s}$ (1) and $\frac{\lambda}{1+r} = 1-\mu$ (2)

Question 15

(i) From (2) $\mu = 1 - \frac{\lambda}{1+r}$ sub into (1)

$$1 - \lambda = \left(1 - \frac{\lambda}{1+r}\right)\frac{s}{1+s}$$

$$\lambda\left(1 - \frac{s}{(1+r)(1+s)}\right) = 1 - \frac{s}{1+s}$$

$$\lambda\,\frac{(1+r)(1+s) - s}{(1+r)(1+s)} = \frac{1+s-s}{1+s}$$

$$\lambda\,\frac{1+s+r+rs-s}{(1+r)(1+s)} = \frac{1}{1+s}$$

$$\lambda\,\frac{(1+r+rs)}{1+r} = 1$$

$$\lambda = \frac{1+r}{1+r+rs}$$

So $\vec{OG} = \left(1 - \frac{1+r}{1+r+rs}\right)\underset{\sim}{a} + \frac{1+r}{1+r+rs} \times \frac{1}{1+r}\,\underset{\sim}{b}$

$$= \frac{1+r+rs-(1+r)}{1+r+rs}\,\underset{\sim}{a} + \frac{1}{1+r+rs}\,\underset{\sim}{b}$$

$$= \frac{rs}{1+r+rs}\,\underset{\sim}{a} + \frac{1}{1+r+rs}\,\underset{\sim}{b}$$

Question 15

(ii) $\vec{OF} = \vec{OA} + \vec{AF}$

$= \underset{\sim}{a} + \frac{t}{t+1}\vec{AB}$

$= \underset{\sim}{a} + \frac{t}{t+1}(\underset{\sim}{b} - \underset{\sim}{a})$

$= \frac{1}{t+1}\underset{\sim}{a} + \frac{t}{t+1}\underset{\sim}{b}$

Also $\vec{OF} = k.\vec{OG}$

$= \frac{krs}{1+r+rs}\underset{\sim}{a} + \frac{k}{1+r+rs}\underset{\sim}{b}$

so $\frac{krs}{1+r+rs} = \frac{1}{t+1}$ ③ and $\frac{k}{1+r+rs} = \frac{t}{t+1}$ ④

Dividing ④ by ③

$t = \frac{k}{1+r+rs} \times \frac{1+r+rs}{krs}$

$t = \frac{1}{rs}$

Question 16

a) (i) Let $u = a - x$ so $x = a - u$

x	0	a
u	a	0

$du = -dx$

$$\int_0^a f(x)\,g(x)\,dx = \int_a^0 f(a-u)\,g(a-u)\,(-du)$$

$$= \int_0^a f(a-u)\,g(a-u)\,du$$

$$= \int_0^a f(a-x)\,g(a-x)\,dx \quad \text{Dummy variable}$$

$$= \int_0^a f(x)\,(k - g(x))\,dx$$

$$= \int_0^a k \cdot f(x)\,dx - \int_0^a f(x)\,g(x)\,dx$$

$$2\int_0^a f(x)\,dx = k\int_0^a f(x)\,dx$$

$$\int_0^a f(x)\,dx = \frac{k}{2}\int_0^a f(x)\,dx$$

(ii) $a = 2$ let $f(x) = |x-1|$

$$f(2-x) = |2-x-1|$$

$$= |1-x|$$

$$= |x-1| = f(x)$$

Question 16

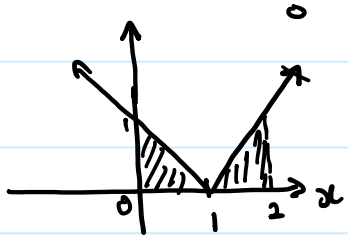
(ii) Let $g(x) = \cos^{-1}(x-1)$

$g(2-x) = \cos^{-1}(2-x-1) = \cos^{-1}(1-x)$

$g(x) + g(2-x) = \cos^{-1}(x-1) + \cos^{-1}(1-x)$

$= \pi$

Using (i) $\int_0^2 |x-1| \cos^{-1}(x-1)\,dx = \frac{\pi}{2} \int_0^2 |x-1|\,dx$



$= \frac{\pi}{2} \times 2 \times \frac{1}{2} \times 1 \times 1$

$= \frac{\pi}{2}$

b) (i) $\omega^k + \omega^{-k} = (\cos\theta + i\sin\theta)^k + (\cos\theta + i\sin\theta)^{-k}$

$= \cos k\theta + i\sin k\theta + \cos(-k\theta) + i\sin(-k\theta)$

$= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta$

$= 2\cos k\theta$

(ii) $S = \omega^{-2n} - \omega^{-2n+1} + \omega^{-2n+2} - \ldots - \omega^{-1} + 1 - \omega + \omega^2 - \ldots - \omega^{2n-1} + \omega^{2n}$

$= \frac{\omega^{-2n}\left(1 - (-\omega)^{4n+1}\right)}{1 - (-\omega)}$ using sum of geometric series

$= \frac{\omega^{-2n}\left(1 + \omega^{4n+1}\right)}{1+\omega} \times \frac{1+\bar{\omega}}{1+\bar{\omega}}$ as $(-\omega)^{4n+1} = -(\omega^{4n+1})$

$= \frac{\left(\omega^{-2n} + \omega^{2n+1}\right)(1+\bar{\omega})}{(1+\omega)(1+\bar{\omega})}$ Now $\bar{\omega} = \frac{1}{\omega}$ as $|\omega| = 1$

$= \frac{\omega^{-2n} + \omega^{-2n-1} + \omega^{2n+1} + \omega^{2n}}{(1+\omega)(1+\bar{\omega})} = \frac{\omega^{-2n} + \omega^{2n} + \omega^{-(2n+1)} + \omega^{2n+1}}{(1+\omega)(1+\bar{\omega})}$

as required.

Question 16

b] (iii) From (ii) $S = \dfrac{\omega^{-2n} + \omega^{2n} + \omega^{-(2n+1)} + \omega^{2n+1}}{1 + \omega + \bar{\omega} + \omega\bar{\omega}}$ Now, $\omega\bar{\omega} = |\omega|^2 = 1$

$$= \frac{2\cos 2n\theta + 2\cos(2n+1)\theta}{1 + 2\cos\theta + 1} \quad \text{using (i)}$$

$$= \frac{2(\cos 2n\theta + \cos(2n+1)\theta)}{2(1+\cos\theta)}$$

$$= \frac{\cos 2n\theta + \cos(2n+1)\theta}{1+\cos\theta}$$

Also by rearranging & grouping terms

$$S = 1 - (\omega + \omega^{-1}) + (\omega^2 + \omega^{-2}) - \ldots - (\omega^{2n-1} + \omega^{-(2n-1)}) + (\omega^{2n} + \omega^{-2n})$$

$$= 1 - 2\cos\theta + 2\cos 2\theta - \ldots - 2\cos(2n-1)\theta + 2\cos 2n\theta$$

$$= 1 - 2(\cos\theta - \cos 2\theta + \ldots + \cos(2n-1)\theta + \cos 2n\theta)$$

Equating the two expressions for S yields the desired result.

(iv) Let $\theta = \frac{\pi}{13}$ and $n = 13$ in (iii)

$$1 - 2\left(\cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \ldots + \cos\frac{25\pi}{13} - \cos\frac{26\pi}{13}\right) = \frac{\cos\frac{26\pi}{13} + \cos\frac{27\pi}{13}}{1 + \cos\frac{\pi}{13}}$$

$$1 - 2\left(\cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \ldots + \cos\frac{25\pi}{13} - 1\right) = \frac{1 + \cos\frac{\pi}{13}}{1 + \cos\frac{\pi}{13}}$$

$$1 - 2\left(\cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \ldots + \cos\frac{25\pi}{13}\right) + 2 = 1$$

$$\cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \ldots + \cos\frac{25\pi}{13} = 1$$

Question 16

c) RTP: there are no non-zero integers a,b,c such that $a^2+b^2+c^2 = 4abc$

Assume for the sake of contradiction that there exist $a,b,c \in \mathbb{Z}\setminus\{0\} : a^2+b^2+c^2 = 4abc$

Firstly note that the RHS is even.

Case 1 – If a,b,c are all odd then a^2, b^2, c^2 are odd and $a^2+b^2+c^2$ is odd. But RHS is even.

Case 2 – If one of them (say a) is odd and b and c are even then a^2 is odd & b^2, c^2 are even. So $a^2+b^2+c^2$ is odd, but RHS is even.

Case 3 – If two of them (say a & b) are odd & c is even a^2, b^2 are odd & c^2 is even. $a^2+b^2+c^2$ is even.

Let $a = 2k+1$ & $b = 2m+1$ & $c = 2p$

$$\text{LHS} = a^2+b^2+c^2 = (2k+1)^2 + (2m+1)^2 + (2p)^2$$
$$= 4k^2+4k+1+4m^2+4m+1+4p^2$$
$$= 4(k^2+k+m^2+m+p^2) + 2$$

which is 2 more than a multiple of 4

RHS $= 4abc$ is a multiple of 4.

Case 4 – If a,b,c are all even a^2, b^2, c^2 are even and $a^2+b^2+c^2$ is even.

Let $a = 2k$, $b = 2m$, $c = 2p$

Then $a^2+b^2+c^2 = 4abc$

$$(2k)^2 + (2m)^2 + (2p)^2 = 4(2k)(2m)(2p)$$
$$4(k^2+m^2+p^2) = 32kmp$$

Question 16

(c) so $k^2+m^2+p^2 = 8kmp$

This is similar to our original result, so if such k, m, p exist they must all be even

let $k = 2k_1$ $m = 2m_1$ and $p = 2p_1$

$$(2k_1)^2 + (2m_1)^2 + (2p_1)^2 = 8(2k_1)(2m_1)(2p_1)$$

$$4(k_1^2 + m_1^2 + p_1^2) = 64(k_1 m_1 p_1)$$

$$k_1^2 + m_1^2 + p_1^2 = 16 k_1 m_1 p_1$$

Again k_1, m_1, p_1 must all be even etc.

This process continues indefinitely implying that a, b, c are able to be divided by 2 repeatedly, infinite times. The only number for which this is true is 0

So $a = b = c = 0$

But a, b, c are non-zero.

Thus, in all cases we arrive at a contradiction. Hence, the statement is true and there are no non-zero integers a, b, c : $a^2+b^2+c^2 = 4abc$.