Student Number: | | | | | | | | |

Teacher Name: _____

Penrith Selective High School

2026 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- Reference sheets are provided with this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 1–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–11)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Q15	Q16	TOTAL
Complex Numbers	1 /1	a, d /6	a /4	a /3		c /5		/19
Vectors	7, 9, 10 /3	b /3	c /5	c /3	b /8			/22
Proof	2, 4 /2	e /3	b, d /6		c /4		c /6	/21
Calculus	3, 6 /2	c /3		e /3	a /3	a /3	a /5	/19
Mechanics	5, 8 /2			b, d /6		b /7	b /4	/19
TOTAL	/10	/15	/15	/15	/15	/15	/15	/100
								%

This paper MUST NOT be removed from the examination room

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 For $z \in C$ and Im $(z) < 0$, then $Arg\left(\frac{z\bar{z}}{z-\bar{z}}\right)$ is given by

A. $\frac{\pi}{2}$

B. $-\frac{\pi}{2}$

C. $\frac{\pi}{4}$

D. 0

2 Which of the following statements is **FALSE**?

A. If a^2 is not divisible by 4, then a is odd.

B. If $m^2 - n^2$ is divisible by 3, then $m - n$ is divisible by 3.

C. If n is any integer, then $n^2 + n$ is even.

D. If $(3x - 3y)$ is even, then x and y are either both odd or both even.

3 Evaluate

$$\int_{-0.5}^{0.5} \frac{dx}{\sqrt{1-2x^2}}$$

A. $\frac{\pi}{\sqrt{2}}$

B. 0

C. $\frac{\sqrt{2}\pi}{2}$

D. $\frac{\pi}{2\sqrt{2}}$

4 Consider the statement:

If $x^2 - 5x + 6 \leq 0$, then $x \geq 2$.

Which of the following is the **contrapositive** of the given statement?

A. If $x^2 - 5x + 6 \leq 0$, then $x < 2$.

B. If $x < 2$, then $x^2 - 5x + 6 > 0$.

C. If $x^2 - 5x + 6 > 0$, then $x < 2$.

D. If $x \geq 2$, then $x^2 - 5x + 6 \leq 0$.

5 A particle moves along the x-axis with acceleration $a = -v(25 + v^2)$.

Intially the particle is at 10 m to the right of the origin with a velocity of -10 metres per second.

Which one of the statements is true?

A. The displacement is given by $-\frac{1}{5}\left(\tan^{-1}\frac{v}{5} - \tan^{-1} 2\right) + 10$.

B. The particle moves in the negative x-direction, slows down and stops after travelling for $-\frac{1}{5}(\tan^{-1} 2)$ metres from its intial position.

C. The particle does not pass through the origin.

D. All of the above is true.

6 Evaluate

$$\int_0^{0.5} \sqrt{\frac{x}{1-x}}\, dx \, .$$

A. $\frac{1}{4} - \frac{\pi}{8}$

B. $\frac{\pi}{4} - \frac{1}{2}$

C. $\frac{1}{2} - \frac{\pi}{4}$

D. $\frac{\pi}{8} - \frac{1}{4}$

7 Given $\underset{\sim}{r} = \begin{pmatrix}3\\4\end{pmatrix}$, $\underset{\sim}{s} = \begin{pmatrix}0\\1\end{pmatrix}$ and $\underset{\sim}{t} = \begin{pmatrix}2\\-2\end{pmatrix}$.

Given O, R, S and T are points on the Cartesian plane with radius vectors $\underset{\sim}{O}, \underset{\sim}{R}, \underset{\sim}{S}, \underset{\sim}{T}$

Find the point(s) F if $\overrightarrow{TF}$ is parallel to $\overrightarrow{SR}$ and twice its length.

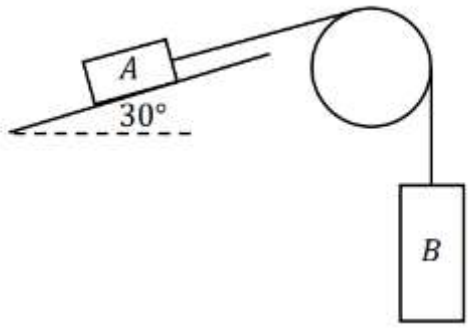
A. $F(4,8)$ or $(-4,-8)$

B. $F(8,4)$ or $(-8,-4)$

C. $F(8,4)$ or $(-4,-8)$

D. $F(4,8)$ or $(-8,-4)$

8 Particle A and B, both same mass m kg, are connected by a light inextensible string passing over a frictionless pulley. Particle A is on a frictionless table inclined at an angle of 30° to the horizontal. Initially both the particles are at rest. After particle A has travelled x metres towards the pulley, it is travelling at v metres per second ($g = 10\frac{m}{s^2}$).



The expression of x in terms of t is given by

A. $x = \frac{5}{2}t^2$

B. $x = \frac{5}{4}t^2$

C. $x = \frac{5}{8}t^2$

D. $x = 5t^2$

9 Consider the vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ where $|\underset{\sim}{r}| = 9$ and $\underset{\sim}{s} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

If the vector projection of $\underset{\sim}{r}$ in the direction of $\underset{\sim}{s}$ is equal to $-4\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$, then the scalar projection of $\underset{\sim}{s}$ in the direction of $\underset{\sim}{r}$ is given by

A. -18

B. -2

C. 3

D. 2

10 Which of the following gives the same curve as $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ t \\ \cos t \end{pmatrix}$?

A. $\begin{pmatrix} \sin\left(\frac{\pi}{2} - t\right) \\ t \\ \cos\left(\frac{\pi}{2} - t\right) \end{pmatrix}$

B. $\begin{pmatrix} \sin t^2 \\ t^2 \\ \cos t^2 \end{pmatrix}$

C. $\begin{pmatrix} \sin 2t \\ 2t \\ \cos\ 2t \end{pmatrix}$

D. $\begin{pmatrix} -\cos\left(2t + \frac{\pi}{2}\right) \\ 2t + \frac{\pi}{2} \\ \sin\left(2t + \frac{\pi}{2}\right) \end{pmatrix}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet for Question 11.

(a) Find real values of x and y in the following expression: **3**

$$\frac{x}{1-2i}+\frac{2y}{2-i}=1-i$$

(b) Find the vector equation of the line l through $P(1,-2,3)$ and parallel to $2\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$. **3**

Hence, determine if the point $Q(7,7,0)$ lies on l.

(c) Evaluate the following: **3**

$$\int_0^1 (4x+1)\sin \pi x \, dx \, .$$

(d) Sketch the region given by $|z|<1$ and $0<\arg(z+i)\leq\frac{\pi}{3}$. Indicate all essential features. **3**

(e) If $m+n$ is even, prove that m^3+n^3 is also even. Is the converse true? Provide a proof or a counter example to justify your response. **3**

End of Question 11

Question 12 (15 marks) Use a separate Writing Booklet for Question 12.

(a) i. Write $\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{k}}$ in $re^{i\theta}$ form. **2**

ii. For what values of k is $\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{k}}$ purely imaginary? **2**

(b) Let a and b are positive real numbers. **3**

Prove that $(a+b)^2 \geq 4ab$, and hence, prove that $\left(\sqrt[3]{x}+\frac{1}{\sqrt[3]{x}}\right)^2 \geq 4$ for all real x except 0.

(c) Define the lines l_1 and l_2 by

$$l_1: \underset{\sim}{r}(\lambda) = \begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$l_2: \underset{\sim}{r}(\mu) = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\4\\3\end{pmatrix}$$

where λ, μ are any real numbers.

i. Prove that l_1 and l_2 are **NOT** skew lines. **2**

ii. Find the equation of a line that passes through the point of intersection of l_1 and l_2 in part (i) and is perpendicular to both lines. **3**

(d) Let $f(x) = \ln x - x + 1$.

i. Show that $\forall x > 0, f(x) \leq 0$. **1**

ii. Let $p_i > 0, \forall i = 1, 2, 3, \ldots, n$ such that $\sum_1^n p_i = 1$. **2**

Prove that $\sum_1^n \ln(np_k) \leq 0$.

End of Question 12

Question 13 (15 marks) Use a separate Writing Booklet for Question 13.

(a) Let $z = x + iy$ represent the square roots of $a + ib$. **3**

Prove the following:

$$x^2 = \frac{1}{2}(|z|^2 + a)$$

and

$$y^2 = \frac{1}{2}(|z|^2 - a)$$

(b) The displacement of a particle moving along the x-axis is given by **3**

$x = 3\sin\left(2t + \frac{\pi}{6}\right) + 2\cos 2t$, where x is in metres and t is in seconds.

Prove that this is a simple harmonic motion and state its amplitude and period.

(c) *Let ABC* be a non-equilateral triangle with position vectors $\overrightarrow{OA}$ $(\underset{\sim}{a})$, $\overrightarrow{OB}$ $(\underset{\sim}{b})$ and **3**
$\overrightarrow{OC}$ $(\underset{\sim}{c})$ relative to the circumcentre O.

Let G be the centroid of the triangle ABC, with position vector $\overrightarrow{OG}$ $(\underset{\sim}{g})$ given by

$$g = \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3}$$

Prove that

$GA^2 + GB^2 + GC^2 = 3(R^2 - OG^2)$, where R is the circumradius.

Question 13 continues on the next page

(d) A lighthouse beacon rotates once every 20 seconds. The light source is mounted on a circular mechanism with a diameter of 4 metres, and the centre of the rotating mechanism is located 1.5 metres above the ground level. **3**

As the light moves in a vertical circular path, the height of the light source above the ground varies in simple harmonic motion.

Prove that the light source is below the ground for $10 - \frac{20}{\pi}\sin^{-1}\left(\frac{3}{4}\right)$ seconds during each complete rotation of the mechanism.

(e) The Laplace Transform is an advanced technique used to solve differential equations. **3**

It is an operational on $f(t)$ defined as

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

Show that $\mathcal{L}(t) = \frac{1}{s^2}$.

End of Question 13

Question 14 (15 marks) Use a separate Writing Booklet for Question 14.

(a) Show that **3**

$$\int_0^{\sqrt{3}} \frac{5(x^2+3)dx}{(x^2-9)(x^2+1)} = \ln\left(2-\sqrt{3}\right) - \frac{\pi}{3}.$$

(b) The base of a pyramid is the parallelogram $ABCD$ with vertices at points $A(2,-1,3), B(4,-2,1), C(a,b,c)$ and $D(4,3,-1)$. The apex of the pyramid is located at $P(4,-4,9)$.

i. Show that the values of a, b and c are given by $6, 2$ and -3 respectively. **1**

ii. Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **1**

iii. Show that the area of the base of the pyramid is given by $2\sqrt{65}\ u^2$. **2**

iv. Show that $6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$, hence find a unit vector that is perpendicular to the base of the pyramid. **2**

v. Show that the height of the pyramid is given by $\frac{36}{\sqrt{65}}\ u$. Hence, find the exact volume of the pyramid. **2**

(c) The Fibonacci sequence is defined by **4**

$$F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2} (n \geq 3).$$

Using the identity

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2},$$

prove by mathematical induction that

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

for all positive integers n.

End of Question 14

Question 15 (15 marks) Use a separate Writing Booklet for Question 15.

(a) Evaluate the following integral: **3**

$$\int_0^1 \frac{x^2 \tan^{-1} x}{x^2+1}\,dx\,.$$

(b) A particle moves in a straight line with acceleration

$$\ddot{x} = -\frac{x^3}{2}$$

Initially, the particle is at rest at $x = 2$.

i. Prove that $v^2 = \frac{1}{4}(16 - x^4)$. **3**

ii. Find the value of the greatest speed, and where it occurs. **3**

iii. Describe the motion of the particle. **1**

(c) Using De Moivre's theorem, prove that **5**

$$\tan 5\theta = \frac{\tan^5\theta - 10\ \tan^3\theta\ + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}.$$

Hence, find the roots of the polynomial $5\tan^4\theta - 10\tan^2\theta + 1 = 0$.

Also, write the exact value of $\tan\frac{\pi}{10}$ in surd form.

End of Question 15

Question 16 (15 marks) Use a separate Writing Booklet for Question 16.

(a) Define for $n \geq 0$

$$I_n = \int_0^1 x(1-x^3)^n dx.$$

i. For $n \geq 1$, show that $I_n = \frac{3n}{3n+2} I_{n-2}$. **3**

ii. Hence, find I_4. **2**

(b) A particle moves in simple harmonic motion given by the model

$$x = A \sin nt.$$

Initially the particle is at the origin and moves towards the positive extremity. Matthew claims that when the particle has displacement $\frac{1}{k}$ of the positive extremity, the velocity is also $\frac{1}{k}$ of the maximum speed, where $k > 1$.

i. Provide a counter example to show that the statement is **NOT** always true. **2**

ii. Find the value of k for which the statement is true. **2**

(c) i. Prove by calculus that $f(s) = \frac{n^2}{s} + \frac{1}{t-s}, t > s > 0,$ has a minimum when $s = \frac{nt}{1+n}$. It is given that $S = \frac{nt}{n-1}$ is a loca maximum of the given function. **2**

ii. Suppose $x_i > 0$ for $i = 1, 2, \ldots, n$ where $n \geq 2$. **4**

Using part (i), show by mathematical induction that

$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq \frac{n^2}{s},$ where $s = x_1 + x_2 + \cdots + x_n$.

(Hint: for $n = k + 1$ case, you may take $t = s + x_{k+1}$.)

END OF PAPER

Multiple Choice

Q1 $\text{Arg}\left(\frac{z\bar{z}}{z-\bar{z}}\right)$, $z\bar{z}=|z|^2$

$z = x+iy$, $\bar{z} = x-iy$

$z-\bar{z} = 2iy$, $y<0$

$= \text{Arg}|z|^2 - \text{Arg}(z-\bar{z})$

$= 0 - (-\pi/2)$

$= \pi/2$ A

Q2 A) The contrapositive method.

a even $\Rightarrow a = 2k$, $k \in \mathbb{Z}$

then $a^2 = 4k^2$, divisible by 4.

True

B) $m^2-n^2 = (m+n)(m-n)$

counter example:

False → $m=1$, $n=2$, $m^2-n^2=-3$

but $m-n=-1$ not divisible by 3

B

c) $n^2+n = n(n+1)$ always

Product of even consecutive integers. True

D) $3(x-y)$ even

$\Rightarrow$ both x, y either odd/even True. True

Q3
$$\int_{-0.5}^{0.5} \frac{dx}{\sqrt{1-2x^2}} = 2\int_{0}^{0.5} \frac{dx}{\sqrt{1-2x^2}}$$

$x=0, \theta=0$

$x=0.5, \theta=\pi/4$

$$= 2\int_{\frac{1}{\sqrt{2}}}^{\pi/4} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{\pi}{2\sqrt{2}}$$ D

Q4 B

Q5 D

Q6
$$\int_0^{0.5} \sqrt{\frac{x}{1-x}}\, dx$$

sub $x = \sin^2\theta$

$$= \int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

$$= \pi/4 - \frac{1}{2}$$ B

Q7 $\vec{TF} \parallel \vec{SR}$, $|\vec{TF}| = 2|\vec{SR}|$, $F = \begin{pmatrix} x \\ y \end{pmatrix}$

$x-2 = \pm 6$

$y+2 = \pm 3\times 2$

$\Rightarrow x = 8, -4$

$y = 6-2, -6-2$

$\vec{TF} = \begin{pmatrix} x-2 \\ y+2 \end{pmatrix}$, $\vec{SR} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$(8,4), (-4,-8)$ C

Q8 $T - mg\sin 30^\circ = ma$, $mg - T = ma$

$\ddot{x} = \frac{1}{4}g$, integrating $\dot{x} = \frac{1}{4}gt$

$x = \frac{5}{4}t^2$ B

Q9 $\text{proj}_{\underset{\sim}{s}} \underset{\sim}{r} = \frac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{\underset{\sim}{s}\cdot\underset{\sim}{s}}\underset{\sim}{s}$, $\text{Proj}_{\underset{\sim}{r}} \underset{\sim}{s} = \frac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{\underset{\sim}{r}\cdot\underset{\sim}{r}}\underset{\sim}{r}$

$|\underset{\sim}{s}| = 3$, sub $\underset{\sim}{w}$ // $\begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix}$

$\underset{\sim}{r}\cdot\underset{\sim}{s} = \frac{-18}{9}$,

$= -2$ B

Q10 C

Question 11 (15 marks)

(a) Find real values of x and y in the following expression: **3**

$$\frac{x}{1-2i}+\frac{2y}{2-i}=1-i$$

$\frac{x}{1-2i}+\frac{2y}{2-i}=1-i$

$\frac{x(1+2i)}{(1-2i)(1+2i)}+\frac{2y(2+i)}{(2-i)(2+i)}=1-i$

$\frac{x(1+2i)+2y(2+i)}{5}=1-i$

$x+4y+(2x+2y)i=5-5i$

equating real and imaginary parts

$x+4y=5 \rightarrow (1)$

$2x+2y=-5 \rightarrow (2)$

From (1): $x=5-4y$

Sub into (2)

$2(5-4y)+2y=-5$

$-6y=-15$

$y=\frac{15}{6}=\frac{5}{2}$

$x=-5$

Marker's Feedback: This question is mostly done well except for occasional error in solving the simultaneous equations.

(b) Find the vector equation of the line l through $P(1,-2,3)$ and parallel to $2\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$. **3**

Hence, determine if the point $Q(7,7,0)$ lies on l.

$\underset{\sim}{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}, \quad \lambda\in\mathbb{R}$

For Q(7,7,0) to lie on l,

$\begin{pmatrix}7\\7\\0\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}$

$1+2\lambda=7 \quad | \quad -2+3\lambda=7 \quad | \quad 3-\lambda=0$

$\lambda=3 \quad | \quad \lambda=\frac{9}{3}=3 \quad | \quad \lambda=3$

$\therefore \lambda$ is consistently $=3$,

$\therefore$ Q(7,7,0) lies on l.

Marker's Feedback: This question is mostly done well.

(c) Evaluate the following: **3**

$$\int_0^1 (4x+1)\sin \pi x \, dx \, .$$

$\int_0^1 (4x+1)\sin \pi x \, dx$

(integration by parts)

let $u = 4x+1$ | $dv = \sin \pi x \, dx$

$du = 4\,dx$ | $v = -\frac{\cos \pi x}{\pi}$

$$I = (4x+1)\left(-\frac{\cos \pi x}{\pi}\right)\Big]_0^1 + \frac{4}{\pi}\int_0^1 \cos \pi x \, dx$$

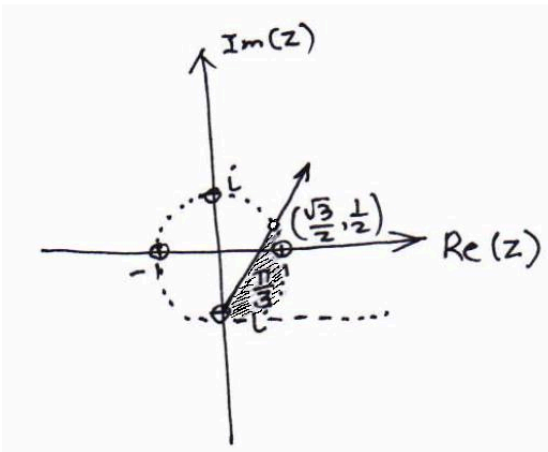
$$I = (4x+1)\left(-\frac{\cos \pi x}{\pi}\right)\Big]_0^1 + \frac{4}{\pi}\left[\frac{\sin \pi x}{\pi}\right]_0^1$$

$$I = \frac{-5\cos \pi}{\pi} - \left(\frac{-\cos 0}{\pi}\right) + \frac{4}{\pi^2}(\sin \pi - \sin 0)$$

$$I = \frac{5}{\pi} + \frac{1}{\pi} + 0 = \frac{6}{\pi}$$

Marker's Feedback: Most common error in this question is the incorrect simplification after substituting values. It is recommended that students state that they are using integration by parts.

(d) Sketch the region given by $|z| < 1$ and $0 < \arg(z+i) \leq \frac{\pi}{3}$. Indicate all essential features. **3**



$x^2 + y^2 = 1 \rightarrow (1)$

$y = (\tan \frac{\pi}{3})x - 1$

$y = \sqrt{3}x - 1 \rightarrow (2)$

Sub (2) into (1)

$x^2 + (\sqrt{3}x - 1)^2 = 1$

$x^2 + 3x^2 - 2\sqrt{3}x + 1 = 1$

$2x(2x - \sqrt{3}) = 0$

$x = 0$ | $x = \frac{\sqrt{3}}{2}$

$y = -1$ | $y = \frac{1}{2}$

Marker's Feedback: 1 mark is allocated for showing the points of intersection between the circle and the vector. Only a couple of students showed this so majority only managed to achieve 2 marks.

(e) If $m + n$ is even, prove that $m^3 + n^3$ is also even. Is the converse true? Provide a proof or a counter example to justify your response. **3**

$m+n$ is even

$\Rightarrow m+n = 2k,\ k \in \mathbb{Z}$

$m^3+n^3 = (m+n)(m^2-mn+n^2)$

$= 2k(m^2-mn+n^2)$

$\therefore m^3+n^3$ is even

Converse:

If m^3+n^3 is even, then $m+n$ is even.

Using contrapositive to prove the converse,

If $m+n$ is odd, then m^3+n^3 is odd.

$m+n$ is odd

$\Rightarrow$ if m is odd, then n is even or vice-versa

i.e. $m = 2p+1,\ n = 2q,\ p, q \in \mathbb{Z}$

$m^3+n^3 = (2p+1)^3 + (2q)^3$

$= 8p^3 + 3(4p^2) + 3(2p) + 1 + 8q^3$

$= 2(4p^3+6p^2+3p+4q^3)+1$

Hence, m^3+n^3 is odd.

$\therefore$ contrapositive is true

$\therefore$ the converse is true

Marker's Feedback: Marks were allocated depending on the accuracy and detail of the proof. A few students failed to mention the converse of the statement. It would be wise to mention the converse before proving whether the converse is true or false.

Question 12 (15 marks)

(a) i. Write $\dfrac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{k}}$ in $re^{i\theta}$ form. **2**

ii. For what values of k is $\dfrac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{k}}$ purely imaginary? **2**

(i) $1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}$

$\sqrt{3}-i = 2e^{-i\frac{\pi}{6}}$

$$\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{k}} = \frac{(2e^{i\frac{\pi}{3}})^{12}}{(2e^{-i\frac{\pi}{6}})^{k}}$$

$$= \frac{2^{12}}{2^{k}} \frac{e^{i4\pi}}{e^{-i\frac{k\pi}{6}}}$$

$$= 2^{12-k} e^{i(4\pi+\frac{k\pi}{6})}$$

Since $e^{i4\pi}=1$

$\therefore 2^{12-k} e^{i\frac{k\pi}{6}}$

(ii) For purely imaginary,

$Re(e^{i\frac{k\pi}{6}}) = 0$

i.e $\cos\frac{k\pi}{6} = 0$

$\frac{k\pi}{6} = \frac{\pi}{2} + n\pi$

multiplying both sides by $\frac{6}{\pi}$

$\boxed{k = 3+6n,\ n\in\mathbb{Z}}$

or $\frac{k\pi}{6} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\frac{k}{6} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$

$k = \pm 3, \pm 9, \pm 15, \ldots$

$\therefore \boxed{k = \pm(6m-3),\ m\in\mathbb{Z}}$

or $4\pi + \dfrac{k\pi}{6} = \dfrac{\pi}{2} + n\pi$

$24\pi + k\pi = 3\pi + 6n\pi$

$24 + k = 3 + 6n$

$\boxed{k = 6n-21, n\in\mathbb{Z}}$

Marker's Feedback: Common errors are students not expressing the final answer in exponential form and forgetting to include the 2 in raising to the power of 12 and k. For part (ii) students are adding $2\pi n$, thus achieving the incorrect answer.

(b) Let a and b are positive real numbers. **3**

Prove that $(a+b)^2 \geq 4ab$, and hence, prove that $\sqrt[3]{x^2} + \frac{1}{\sqrt[3]{x^2}} \geq 2$ for all real x except 0.

$(a-b)^2 \geq 0$, $a, b \in \mathbb{R}$

$a^2+b^2-2ab \geq 0$

$a^2+b^2 \geq 2ab$

$a^2+2ab+b^2 \geq 4ab$

$(a+b)^2 \geq 4ab$

Now let $a = \sqrt[3]{x}$, $b = \frac{1}{\sqrt[3]{x}}$, where $x \neq 0$

$$\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4\sqrt[3]{x}\left(\frac{1}{\sqrt[3]{x}}\right)$$

$$\sqrt[3]{x^2} + 2 + \frac{1}{\sqrt[3]{x^2}} \geq 4$$

$$\therefore \sqrt[3]{x^2} + \frac{1}{\sqrt[3]{x^2}} \geq 2$$

Marker's Feedback: This question was mostly done well. Common error includes incorrect substitution for a and b.

(c) Define the lines l_1 and l_2 by

$$l_1: \underset{\sim}{r}(\lambda) = \begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix} \qquad l_2: \underset{\sim}{r}(\mu) = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\4\\3\end{pmatrix}$$

where λ, μ are any real numbers.

i. Prove that l_1 and l_2 are **NOT** skew lines. **2**

ii. Find the equation of a line that passes through the point of intersection of l_1 and l_2 in part (i) and is perpendicular to both lines. **3**

(i) For not skew, l_1 and l_2 should intersect

i.e. $\begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\4\\3\end{pmatrix}$

$2+\lambda = 2+\mu \rightarrow ①$

$1+2\lambda = -1+4\mu \rightarrow ②$

$1+\lambda = -1+3\mu \rightarrow ③$

From ①:

$\lambda = \mu$

Sub into ②

$1+2\mu = -1+4\mu$

$2 = 2\mu$

$\therefore \mu = 1, \lambda = 1$

Check if it works with eq ③

$1+\lambda = -1+3\mu$

LHS $= 1+1 = 2$

RHS $= -1+3(1) = 2$

LHS = RHS

$\therefore l_1$ intersect with l_2 at $\begin{pmatrix}3\\3\\2\end{pmatrix}$.

(ii) let l_3 be the eq. of a line that passes through $(3,3,1)$ and $l_1 \perp l_3$ and $l_2 \perp l_3$.

$l_3: \underset{\sim}{r}(t) = \begin{pmatrix}3\\3\\1\end{pmatrix} + t\begin{pmatrix}x\\y\\z\end{pmatrix}$

For perpendicular lines, dot product $= 0$.

$\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \Rightarrow x+2y+z = 0 \rightarrow ④$

$\begin{pmatrix}1\\4\\3\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \Rightarrow x+4y+3z = 0 \rightarrow ⑤$

⑤ − ④

$2y+2z = 0$

$\Rightarrow y = -z$

Sub into ④

$x+2(-z)+z = 0$

$\boxed{x = z}$

$l_3: \underset{\sim}{r}(t) = \begin{pmatrix}3\\3\\2\end{pmatrix} + t\begin{pmatrix}z\\-z\\z\end{pmatrix}$

$\underset{\sim}{r}(t) = \begin{pmatrix}3\\3\\2\end{pmatrix} + tz\begin{pmatrix}1\\-1\\1\end{pmatrix}$

$\underset{\sim}{r}(s) = \begin{pmatrix}3\\3\\2\end{pmatrix} + s\begin{pmatrix}1\\-1\\1\end{pmatrix}$

where $s \in R$

Marker's Feedback: This question was mostly done well, except for some students who did not solve the dot product of the given direction vectors.

(d) Let $f(x) = \ln x - x + 1$.

i. Show that $\forall x > 0, f(x) \leq 0$. **1**

ii. Let $p_i > 0, \forall i = 1, 2, 3, \ldots, n$ such that $\sum_1^n p_i = 1$. **2**

Prove that $\sum_1^n \ln(np_k) \leq 0$.

(i) $f(x) = \ln x - x + 1$

$\forall x > 0$, show that $f(x) \leq 0$

$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$ | $f''(x) = \frac{-1}{x^2}$

Stationary pt. when $f'(x) = 0$

$\frac{1-x}{x} = 0$

$\therefore x = 1$, $f(x) = \ln x - x + 1$

$f(1) = 0$

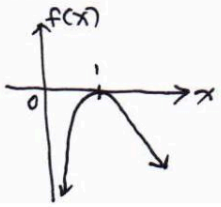
Stationary pt. at $(1, 0)$

$f''(1) = \frac{-1}{1^2} = -1 \Rightarrow$ concave down

$\therefore (1, 0)$ is a maximum

x	0.5	1	2
$f(x)$	1	0	-0.5
	/	—	\

$\therefore \forall x > 0, f(x) \leq 0$.



(ii) $\sum_1^n p_i = 1$

$\Rightarrow p_1 + p_2 + p_3 + \ldots + p_n = 1$

Prove that $\sum_1^n \ln(np_k) \leq 0$

from (i), $\forall x > 0$, $\ln x - x + 1 \leq 0$

let $x = np_k$

$\ln(np_k) - np_k + 1 \leq 0$

$\ln(np_k) \leq np_k - 1$

$\Rightarrow \sum_1^n \ln(np_k) \leq \sum_1^n (np_k - 1)$

RHS $= n\sum_1^n p_k - n$

$= n(1) - n$ (given)

$= 0$

$\therefore \sum_1^n \ln(np_k) \leq 0$.

Marker's Feedback: For part (i), since it is a one-mark question, the marks allocated depended on the marker's discretion. Students who showed knowledge of first and second derivatives were likely to achieve the mark. For part (ii), majority of the students only achieved 1 mark. Students should show evidence that they can link part (ii) with part (i) and demonstrate correct application of the sigma notation.

Question 13 (15 marks)

(a) Let $z = x + iy$ represent the square roots of $a + ib$. **3**

Prove the following:

$$x^2 = \frac{1}{2}(|z^2| + a) \quad \text{and} \quad y^2 = \frac{1}{2}(|z^2| - a)$$

$z = x + iy$

$z^2 = a + ib$

Since $(x + iy)^2 = x^2 - y^2 + 2xy\,i$,

equating real and imaginary parts gives

$$x^2 - y^2 = a \qquad (1)$$

$$2xy = b \qquad (2)$$

Also, $|z|^2 = x^2 + y^2$. (3)

Adding (1) and (3),

$2x^2 = |z|^2 + a$,

$\boxed{x^2 = \frac{1}{2}(|z|^2 + a)}$.

Subtracting (1) from (3),

$2y^2 = |z|^2 - a$,

$\boxed{y^2 = \frac{1}{2}(|z|^2 - a)}$.

Marker's Feedback: This question was mostly done well in various ways. Marks were given depending on the accuracy of the proof.

(b) The displacement of a particle moving along the x-axis is given by **3**

$x = 3\sin\left(2t + \frac{\pi}{6}\right) + 2\cos 2t$, where x is in metres and t is in seconds.

Prove that this is a simple harmonic motion and state its amplitude and period.

$x = 3\sin\left(2t + \frac{\pi}{6}\right) + 2\cos 2t$

$x = 3\sin 2t\cos\frac{\pi}{6} + 3\cos 2t\sin\frac{\pi}{6} + 2\cos 2t$

$x = \frac{3\sqrt{3}}{2}\sin 2t + \frac{3}{2}\cos 2t + 2\cos 2t$

$x = \frac{3\sqrt{3}}{2}\sin 2t + \frac{7}{2}\cos 2t$

$\dot{x} = 3\sqrt{3}\cos 2t - 7\sin 2t$

$\ddot{x} = -6\sqrt{3}\sin 2t - 14\cos 2t$

$\ddot{x} = -4\left(\frac{3\sqrt{3}}{2}\sin 2t + \frac{7}{2}\cos 2t\right)$

$\ddot{x} = -4x$

acceleration is in the form of $\ddot{x} = -n^2x$

$\therefore$ SHM.

Amplitude $= \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{7}{2}\right)^2}$

$= \sqrt{19}$ m

Period $= \frac{2\pi}{2}$ s $= \pi$ s

Marker's Feedback: A few students forgot to show why it is SHM. Common error is not being able to solve for the amplitude.

(c) *Let ABC* be a non-equilateral triangle with position vectors $\overrightarrow{OA}$ $(\underset{\sim}{a})$, $\overrightarrow{OB}$ $(\underset{\sim}{b})$ and $\overrightarrow{OC}$ $(\underset{\sim}{c})$ relative to the circumcentre O. **3**

Let G be the centroid of the triangle ABC, with position vector $\overrightarrow{OG}$ $(\underset{\sim}{g})$ given by

$$\underset{\sim}{g} = \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3}$$

Prove that

$GA^2 + GB^2 + GC^2 = 3(R^2 - OG^2)$, where R is the circumradius.

Given: $\underset{\sim}{g} = \frac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}$

$\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c} = 3\underset{\sim}{g}$ as O is the circumcentre

also $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = R$ (circumradius)

$GA^2 = (\underset{\sim}{a}-\underset{\sim}{g})\cdot(\underset{\sim}{a}-\underset{\sim}{g})$

$GB^2 = (\underset{\sim}{b}-\underset{\sim}{g})\cdot(\underset{\sim}{b}-\underset{\sim}{g})$

$GC^2 = (\underset{\sim}{c}-\underset{\sim}{g})\cdot(\underset{\sim}{c}-\underset{\sim}{g})$

$GA^2+GB^2+GC^2 = |\underset{\sim}{a}|^2+|\underset{\sim}{g}|^2-2\underset{\sim}{a}\cdot\underset{\sim}{g}$

$+ |\underset{\sim}{b}|^2+|\underset{\sim}{g}|^2-2\underset{\sim}{b}\cdot\underset{\sim}{g}$

$+ |\underset{\sim}{c}|^2+|\underset{\sim}{g}|^2-2\underset{\sim}{c}\cdot\underset{\sim}{g}$

$= |\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+3|\underset{\sim}{g}|^2-2(\underset{\sim}{a}\cdot\underset{\sim}{g}+\underset{\sim}{b}\cdot\underset{\sim}{g}+\underset{\sim}{c}\cdot\underset{\sim}{g})$

$= R^2+R^2+R^2+3|\underset{\sim}{g}|^2-2\underset{\sim}{g}\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})$

$= 3R^2+3|\underset{\sim}{g}|^2-2\underset{\sim}{g}\cdot 3\underset{\sim}{g}$

$= 3R^2+3|\underset{\sim}{g}|^2-6|\underset{\sim}{g}|^2$

$= 3R^2-3|\underset{\sim}{g}|^2$

$= 3(R^2-OG^2)$

$\therefore GA^2+GB^2+GC^2 = 3(R^2-OG^2)$

Marker's Feedback: In order to achieve full marks, students should demonstrate the correct use of the dot product and the distributive law. Only a few students achieved full marks for this question.

(d) A lighthouse beacon rotates once every 20 seconds. The light source is mounted on a circular mechanism with a diameter of 4 metres, and the centre of the rotating mechanism is located 1.5 metres above the ground level. **3**

As the light moves in a vertical circular path, the height of the light source above the ground varies in simple harmonic motion.

Prove that the light source is below the ground for $10 - \frac{20}{\pi}\sin^{-1}\left(\frac{3}{4}\right)$ seconds during each complete rotation of the mechanism.

Let h be the height of the light above the ground, in metres.

The circular mechanism has diameter 4 m, so its radius is 2 m.

The centre is 1.5 m above the ground, so the height can be modelled by

$$h = 1.5 + 2\sin\theta,$$

where θ is the angle through which the mechanism has rotated.

Since one complete rotation takes 20 seconds,

$$\omega = \frac{2\pi}{20} = \frac{\pi}{10},$$

so we may write $\boxed{h = 1.5 + 2\sin\left(\frac{\pi t}{10}\right)}$

The light is below the ground when $h < 0$.

Thus $1.5 + 2\sin\left(\frac{\pi t}{10}\right) < 0,$

$$\sin\left(\frac{\pi t}{10}\right) < -\frac{3}{4}.$$

sine is negative in quadrants 3 and 4

Let $\alpha = \sin^{-1}\left(\frac{3}{4}\right)$.

Hence the angular interval during which the light is below the ground is

$$(2\pi - \alpha) - (\pi + \alpha) = \pi - 2\alpha.$$

Since a full angle of 2π corresponds to 20 seconds, the required time is

$$T = \frac{\pi - 2\alpha}{2\pi} \times 20.$$

$$T = 10 - \frac{20\alpha}{\pi}.$$

Substituting $\alpha = \sin^{-1}\left(\frac{3}{4}\right)$,

$$\boxed{T = 10 - \frac{20}{\pi}\sin^{-1}\left(\frac{3}{4}\right) \text{ seconds}}$$

Marker's Feedback: This question was very poorly done.

(e) The Laplace Transform is an advanced technique used to solve differential equations. **3**

It is an operational on $f(t)$ defined as

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

Show that $\mathcal{L}(t) = \frac{1}{s^2}$.

$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

Show that $\mathcal{L}(t) = \frac{1}{s^2}$

$\mathcal{L}(t) = \int_0^\infty e^{-st}\, t\, dt$

by integration by parts

$u = t \quad | \quad dv = e^{-st} dt$

$du = dt \quad | \quad v = \frac{e^{-st}}{-s}$

$\mathcal{L}(t) = t\frac{e^{-st}}{-s}\Big]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt$

$= 0 - 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{1}{-s^2}[0 - 1]$

$\mathcal{L}(t) = \frac{1}{s^2}$

Marker's Feedback: Students who recognised that they simply must replace $f(t)$ with t and used integration by parts were likely able to achieve full marks.

Q14 Show that

a) $\int_0^{\sqrt{3}} \frac{5(x^2+3)\,dx}{(x^2-9)(x^2+1)} = \ln(2-\sqrt{3}) - \pi/3$

$$\frac{5(x^2+3)}{(x^2-9)(x^2+1)} \equiv \frac{A}{x+3} + \frac{B}{x-3} + \frac{Cx+D}{x^2+1}$$

$$5(x^2+3) \equiv A(x-3)(x^2+1) + B(x+3)(x^2+1) + (x+D)(x^2-9)$$

$x=3$, $60 = 60B$, $\boxed{B=1}$

$x=-3$, $60 = -60A$, $\boxed{A=-1}$

$x=0$, $15 = -3A+3B-9D$, $9D=-9$, $\boxed{D=-1}$

$x=1$, $20 = -4A+8B+(-8)(C+D)$, $8 = -8(C+D)$, $C+D=-1$, $\boxed{C=0}$

$$\therefore I = \int_0^{\sqrt{3}} \frac{-dx}{x+3} + \int_0^{\sqrt{3}} \frac{dx}{x-3} + \int_0^{\sqrt{3}} \frac{-1}{x^2+1}\,dx$$

$$= -\left[\ln|x+3|\right]_0^{\sqrt{3}} + \left[\ln|x-3|\right]_0^{\sqrt{3}} - \left[\tan^{-1}x\right]_0^{\sqrt{3}}$$

$$= -\ln|\sqrt{3}+3| + \ln 3 + \ln|\sqrt{3}-3| - \ln 3 - \pi/3$$

$$= \ln\left|\frac{\sqrt{3}-3}{\sqrt{3}+3}\right| - \pi/3$$

$$= \ln|2-\sqrt{3}| - \pi/3$$

$= \text{RHS}$ Hence, Proved

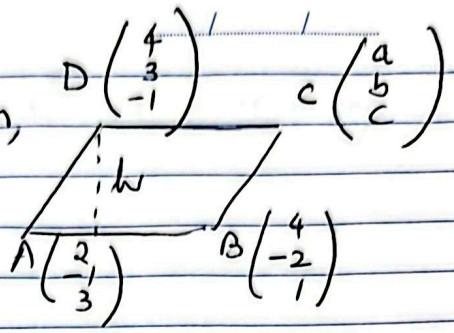
$$\frac{3-\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{9+3-6\sqrt{3}}{9-3} = \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}$$

Comments: Many students did not rationalise (straight wrote the read answer)

- Some students divided the integrand into $\frac{Ax+B}{x^2-9} + \frac{Cx+D}{x^2+1}$ and then divided it further.

Please try to avoid this method, as it is very long. Overall, okay.

b) As ABCD is a parallelogram, opposite sides are parallel and equal in length.



(i) $\vec{AB} = \begin{pmatrix}4-2\\-2+1\\1-3\end{pmatrix}$, $\vec{DC} = \begin{pmatrix}a-4\\b-3\\c+1\end{pmatrix}$

well done !!

$\vec{AB} = \vec{DC}$

$\begin{pmatrix}2\\-1\\-2\end{pmatrix} = \begin{pmatrix}a-4\\b-3\\c+1\end{pmatrix} \Rightarrow \begin{matrix}a = 4+2 \\ b = 3-1 \\ c = -2-1\end{matrix} \begin{matrix}=6\\=2\\=-3\end{matrix}$

(ii) $\vec{AD} = \begin{pmatrix}4-2\\3+1\\-1-3\end{pmatrix}$ $\quad \vec{AB} \cdot \vec{AD} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \cdot \begin{pmatrix}2\\4\\-4\end{pmatrix}$

$= 2\times2 + 4\times-1 + -2\times-4$

$= 8$

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

$8 = |\vec{AB}||\vec{AD}| \cos\theta$

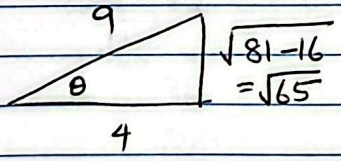
$\therefore 8 = 3\times6\cos\theta$

$\cos\theta = \frac{8}{18} \left(= \frac{4}{9}\right)$

$|\vec{AB}| = \sqrt{2^2+1^2+1^2} = 3$, $\vec{AD} = \sqrt{2^2+4^2+4^2} = 6$

well done !!

(iii) Area of a parallelogram: base × height



$= |\vec{AB}| \times h$

well done !!

$= |\vec{AB}||\vec{AD}| \sin\theta$

$= 3 \times \frac{\sqrt{65}}{9} \times 6$

$= 2\sqrt{65}$ unit^2

(iv) $6\hat{i} + 2\hat{j} + 5\hat{k}$ is $\perp$ $\vec{AB}$ and $\vec{AD}$

$\Rightarrow \begin{pmatrix}6\\2\\5\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\-2\end{pmatrix} = 0$

LHS $6\times2 + 2\times-1 + 5\times-2$

$= 12 - 2 - 10$

$= 0$

$= RHS$

True.

Similarly, $\begin{pmatrix}6\\2\\5\end{pmatrix} \cdot \begin{pmatrix}2\\4\\-4\end{pmatrix}$

$= 12 + 8 - 20$

$= 0$

well done !!

$\therefore \begin{pmatrix}6\\2\\5\end{pmatrix} \perp \vec{AD}$

unit vector $\perp$ to $\vec{AB}$ and $\vec{AD}$ is given by

$$\hat{p} = \frac{6\hat{i}+2\hat{j}+5\hat{k}}{\sqrt{6^2+2^2+5^2}}$$

$$= \frac{6\hat{i}+2\hat{j}+5\hat{k}}{\sqrt{65}}$$

well done !!

$$= \frac{1}{\sqrt{65}}\begin{pmatrix}6\\2\\5\end{pmatrix}$$

v) Volume of the Pyramid $= \frac{1}{3} A \times h$.

h = height is the component of $\vec{AP}$ in the normal direction

$= |\vec{AP} \cdot \hat{p}|$

$$= \left|\frac{1}{\sqrt{65}}\begin{pmatrix}6\\2\\5\end{pmatrix} \cdot \begin{pmatrix}4\\3\\2\end{pmatrix}\right|$$

$$= \frac{1}{\sqrt{65}}(24+6+10)$$

$$= \frac{1}{\sqrt{65}}\left|\begin{pmatrix}2\\-3\\6\end{pmatrix} \cdot \begin{pmatrix}6\\2\\5\end{pmatrix}\right|$$

$$= \frac{1}{\sqrt{65}}|12-6+30|$$

$$= \frac{36}{\sqrt{65}}$$

$$\vec{AP} = \begin{pmatrix}6-2\\-4--1\\9-3\end{pmatrix} = \begin{pmatrix}4\\-3\\6\end{pmatrix}$$

comments :
- Many students were struggling to find the correct height
- Overall okay !!

$$\text{Volume} = \frac{1}{3} \times 2\sqrt{65} \times \frac{36}{\sqrt{65}}$$

$$= 24 \text{ u}^2$$

Q14 c) $F_1 = F_2 = 1$, $F_3 = F_1 + F_2$ $(n \geq 3)$

$= 1 + 1$

$= 2$

$F_4 = F_3 + F_2$

$= 3$

- students were only proving for $n=1$, they forgot they have to prove all the basic platforms.

$F_1 = \frac{\alpha - \beta}{\sqrt{5}}$, $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$

$= \frac{2\sqrt{5}}{2\sqrt{5}}$

$\therefore \boxed{F_1 = 1}$, $F_1^2 = 1$, $F_1 F_2 = 1 \times 1$

$= 1$

$F_2 = \left(\frac{\alpha^2 - \beta^2}{\sqrt{5}}\right)$

$= \left(\frac{\alpha - \beta}{\sqrt{5}}\right)(\alpha + \beta)$

$= 1\left(\frac{1}{2} + \frac{1}{2}\right)$, $F_1^2 + F_2^2 = 1^2 + 1^2$, $F_2 F_3$

$= 1$ $= 2$ $= 1 \times 2$

$= 2$

$\therefore F_1^2 = F_1 F_2$ (result is true for $n=1$)

$F_1^2 + F_2^2 = F_2 F_3$

(the result is true for $n=2$)

$\boxed{F_2 = 1}$

$F_3 = \left(\frac{\alpha^3 - \beta^3}{\sqrt{5}}\right)$

$= \left(\frac{\alpha - \beta}{\sqrt{5}}\right)(\alpha^2 + \alpha\beta + \beta^2)$

$= 1\left(\frac{1+5+2\sqrt{5}}{4} + \frac{1-5}{4} + \frac{1+2\sqrt{5}+5}{4}\right)$

$= 2$

the result is true for $n=3$

$F_4 = \left(\frac{\alpha^4 - \beta^4}{\sqrt{5}}\right)$

$= \left(\frac{\alpha^2 - \beta^2}{\sqrt{5}}\right)(\alpha^2 + \beta^2)$

$= \left(\frac{\alpha - \beta}{\sqrt{5}}\right)(\alpha + \beta)(\alpha^2 + \beta^2)$

$= 1 \times 1 \times \left(\frac{1+5+2\sqrt{5}+1+5-2\sqrt{5}}{4}\right)$

$= 3$

$F_1^2 + F_2^2 + F_3^2$, $F_3 F_4$

$= 1 + 1 + 4$ $= 3 \times 2$

$= 6$ $= 6$

(the result is true for $n=3$)

Let the result be true for $n=k$

$$F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$$

to show the result is true for $n = k+1$

ie. $F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2}$

LHS: $F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2$

$= F_k F_{k+1} + F_{k+1}^2$ (using assumption)

$= F_{k+1}(F_k + F_{k+1})$

$= F_{k+1} F_{k+2}$ (given relationship)

$=$ RHS

Hence using Principle of mathematical induction, the result is true $\forall n \geq 1$

comments: many students were successful in proving the case $n=k+1$.

- students need to improve on the writing format for induction questions.
e.g. students need to avoid the sentence "~~[illegible]~~ $n=1$", Let $n=k$, then $n=k+1$"

Q15

a) $I = \int_0^1 \frac{x^2 \tan^{-1} x}{1+x^2}\, dx$

$= \int_0^1 \frac{(1+x^2-1)\tan^{-1} x}{1+x^2}\, dx$

$= \int_0^1 1 \cdot \tan^{-1} x\, dx - \int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

Let $A = \tan^{-1} x$

$\frac{dA}{dx} = \frac{1}{1+x^2}$

$= \left[\tan^{-1} x \cdot x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1}{1+x^2} 2x\, dx - \int_0^{\pi/4} A\, dA$

$= \pi/4 - 0 - \frac{1}{2}\left[\ln|1+x^2|\right]_0^1 - \left[\frac{A^2}{2}\right]_0^{\pi/4}$

$= \pi/4 - \frac{1}{2}\ln 2 + \ln 1 - \frac{\pi^2}{32}$

$= \pi/4 - \frac{1}{2}\ln 2 - \frac{\pi^2}{32}$

comments :- many students arrive at the correct answer

- common mistakes involve simplifying algebraic equations
- overall, okay

b) $\ddot{x} = -\frac{x^3}{2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{x^3}{2}$

(i) $\frac{1}{2}v^2 = \frac{-x^4}{4\times 2} + c$

at $x = 2$, $v = 0$

well done !!

$0 = -\frac{2^4}{8} + c$

$\Rightarrow c = 2$

$\therefore \frac{1}{2}v^2 = -\frac{x^4}{8} + 2$

$\frac{1}{2}v^2 = \frac{1}{8}(16 - x^4)$

$v^2 = \frac{1}{4}(16 - x^4)$

(ii) the greatest speed occurs at $x = 0$

$\therefore v^2 = \frac{1}{4}(16 - 0)$

well done.

$v^2 = 4$

$v = \pm 2$

$\therefore |v| = 2$ max speed 2 m/s at $x = 0$.

(iii) the particle starts from rest at $x = 2$ moves to the left, accelerating towards the origin, then slowing down until $x = -2$.

comments: students need to learn how to write a question involving "describing the motion".

- You need to focus on starting point, velocity, slowing/increasing and where it is heading.

c) let $z = \cos\theta + i\sin\theta$

$z^5 = (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i\sin 5\theta$ (D' Moivre theorem)

Also, $z^5 = \binom{5}{0}\cos^5\theta + \binom{5}{1}\cos^4\theta\,(i\sin\theta) + \binom{5}{2}\cos^3\theta\,(i\sin\theta)^2 + \binom{5}{3}(i\sin\theta)^3\cdot\cos^2\theta + \binom{5}{4}(i\sin\theta)^4\cdot\cos\theta + \binom{5}{5}(i\sin\theta)^5$

$= 5\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\sin^4\theta\cos\theta$

$+ i(5\cos^4\theta\sin\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta)$

equating real and imaginary components of both the expressions,

$\cos 5\theta = 5\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\sin^4\theta\cos\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta$

dividing $\left(\tan A = \frac{\sin A}{\cos A}\right)$

$$\tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta \;\div \cos^5\theta}{5\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\sin^4\theta\cos\theta \;\div \cos^5\theta}$$

$$= \frac{5\frac{\sin\theta}{\cos\theta} - 10\frac{\sin^3\theta}{\cos^3\theta} + \frac{\sin^5\theta}{\cos^5\theta}}{5 - 10\frac{\sin^2\theta}{\cos^2\theta} + 5\frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{5 - 10\tan^2\theta + 5\tan^4\theta}$$

Hence, proved

Many students were successful in proving this relationship. However, a number of algebra mistakes were evident in few cases. – overall okay.

$5\tan^4\theta - 10\tan^2\theta + 1 = 0$

$\Rightarrow$ $\tan 5\theta$ is undefined (using the above relation)

$5\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots\ldots$

$\theta = \pm\frac{\pi}{10}, \pm\frac{3\pi}{10}, \pm\frac{5\pi}{10}, \pm\frac{7\pi}{10}, \ldots$

However, the only desired solns are

$\pm\frac{\pi}{10}, \pm\frac{3\pi}{10}$, $\frac{5\pi}{10} = \frac{\pi}{2}$, $\tan\frac{\pi}{2}$ is undefd so, it is not a soln)

or

$\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Also, $5\tan^4\theta - 10\tan^2\theta + 1 = 0$ is a reducible quadratic

$5U^2 - 10U + 1 = 0$, $U = \tan^2\theta$

$U = \frac{10 \pm \sqrt{100 - 20}}{10}$

$= \frac{10 \pm 4\sqrt{5}}{10}$

$U = \frac{5 \pm 2\sqrt{5}}{5}$

$\tan^2\theta = \frac{5 \pm 2\sqrt{5}}{5}$

$\tan\theta = \pm\sqrt{\frac{5 \pm 2\sqrt{5}}{5}}$

$\tan\frac{\pi}{10} = \sqrt{\frac{5 - 2\sqrt{5}}{5}}$

as $\frac{\pi}{10}$ is an acute angle.

comments:

- students need to practice questions like this (writing desired solutions)
- students are strongly advised not to use general formula as it leads to some misconception, evident in students work
- overall, just okay.

Q16.

a) $\int_0^1 x(1-x^3)^n dx$.

(i) $\forall n \geq 1$, show that $I_n = \frac{3n}{3n+2} I_{n-1}$

$$I_n = \int_0^1 x(1-x^2)^n dx$$

$$= \left[(1-x^2)^n \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 3n(1-x^3)^{n-1} \cdot -2x \cdot \frac{x^2}{2} dx \quad \text{(by parts)}$$

$$= [0-0] + 3n \int_0^1 (1-x^3)^{n-1} (x^3) x \, dx$$

$$= 3n \int_0^1 (1-x^3)^{n-1} (-(1-x^3)+1) x \, dx$$

$$= -3n \int_0^1 (1-x^3)^n x \, dx + n \int_0^1 (1-x^3)^{n-1} \cdot x \, dx$$

$$I_n = -3n I_n + 3n I_{n-1}$$

$$I_n + 3n I_n = 3n I_{n-1}$$

$$\left(\frac{3n+2}{2}\right) I_n = \frac{3n}{2} I_{n-1}$$

$$\Rightarrow I_n = \frac{3n}{3n+2} I_{n-1}$$

comments:
well done!!

- very few students messed up.

ii) $I_4 = \frac{3(4)}{3(4)+2} I_3$

$$= \frac{12}{14} \times \frac{3(3)}{3(3)+2} I_2$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{3(2)}{3(2)+2} I_1$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{3+2} I_0$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \times \frac{1}{2}$$

$$= \frac{243}{1540}$$

$$I_0 = \int_0^1 x \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

comments:
- few students wrote $I_0 = 1$ instead of $1/2$.
- overall, well done.

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Q16 b)

(i)

$x = A\sin nt$

$\dot{x} = An\cos nt$

$\ddot{x} = -An^2 \sin nt$

$|x| = A, \quad |\dot{x}| = An$

when $x = \frac{A}{k}, \quad k > 1$

$\sin nt = \frac{1}{k}$

$\sin^2 nt + \cos^2 nt = 1$

$\cos nt = \sqrt{1 - \frac{1}{k^2}}$

particle is moving towards positive extremity, $0 < nt < \frac{\pi}{2}$

counter example:

$k = 2, \quad x = \frac{A}{2}$

$\dot{x} = \frac{A}{n}\sqrt{1 - \frac{1}{4}}$

$= \frac{\sqrt{3}}{2}\frac{A}{n}$

$\neq \frac{A}{2}$

False

(ii) $\dot{x} = \frac{1}{k} An$

$An\sqrt{1 - \frac{1}{k^2}} = \frac{1}{k} An$

$\Rightarrow 1 - \frac{1}{k^2} = \frac{1}{k^2}$

$= \frac{2}{k^2} = 1 \Rightarrow k = \sqrt{2} \quad (k > 1)$

comments:

many students found $\dot{x}$, $\ddot{x}$ and put the conditions in as well, but inconsistent approach was evident

- overall okay.

→ well done ($k = 2$)

- many students found the k value, but forgot $k > 1$, overall, okay!!

Question 16 c)

$f(s) = \frac{n^2}{s} + \frac{1}{t-s}, \quad t > s > 0$

$f'(s) = -\frac{n^2}{s^2} - \frac{1}{(t-s)^2} \times -1$

for turning points, $f'(s) = 0$

$\frac{-n^2}{s^2} + \frac{1}{(t-s)^2} = 0$

$\Rightarrow \frac{n^2}{s^2} = \frac{1}{(t-s)^2}$

$\Rightarrow \frac{n}{s} = \pm \frac{1}{t-s}$

$\Rightarrow nt - ns = s$ or $-nt + ns = s$

$nt = s(1+n)$ or $s(n-1) = nt$

$s = \frac{nt}{n+1}$ $\qquad$ $s = \frac{nt}{n-1}$ ↓ 2

to show min/max,

$f''(s) = \frac{2n^2}{s^3} - \frac{2}{(t-s)^3} \times -1$

$= \frac{2n^3}{s^3} + \frac{2}{(t-s)^3}$

at $s = \frac{nt}{n+1}$

$f''(s) = \frac{2n^3(n+1)^3}{n^3t^3} + \frac{2}{\left(t - \frac{nt}{n+1}\right)^3}$

$= \frac{2(n+1)^3}{t^3} + \frac{2(n+1)^3}{t^3}$

$= 4\left(\frac{n+1}{t}\right)^3 > 0.$

$\therefore s = \frac{nt}{n+1}$ is a min.

no need to show anything for this as it is not been asked in the question

(ii) $S = x_1 + \cdots + x_n$.

Step 1 for $n = 2$,

to show,

$\frac{1}{x_1} + \frac{1}{x_2} \geq \frac{4}{S}$

$\Rightarrow \frac{x_1 + x_2}{x_1 x_2} \geq \frac{4}{x_1 + x_2}$

$(a+b)^2 - 4ab \geq 0$

$(a-b)^2 \geq 0$

$a^2 + b^2 \geq 2ab$.

enough to prove

$(x_1 + x_2)^2 \geq 4x_1x_2$

enough to prove

$(x_1 + x_2)^2 - 4x_1x_2 \geq 0$

enough to prove

$(x_1 - x_2)^2 \geq 0$

which is true.

∴ the result is true for $n = 2$.

Step 2 Let the result be true for $n = k$

$\therefore \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \geq \frac{k^2}{S}$

Step 3 To prove that the result is true for $n = k+1$

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(22)

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}} \geq \frac{(k+1)^2}{s}$$

LHS: $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}$

$\geq \frac{k^2}{s}+\frac{1}{x_{k+1}}$ (from the assumption)

(by letting $t = s + x_{k+1}$

$= \frac{k^2}{s}+\frac{1}{t-s}$

$= f(s)$

$\geq \frac{(k+1)^2}{t}$ as it is the minimum value.

Hence proved.

using Principle of Mathematical induction, the result is true $\forall n \geq 2$.

from part (i)

$f(s=\frac{nt}{n+1})$ (min Val)

$= \frac{n^2}{nt/n+1}+\frac{1}{t-\frac{nt}{n+1}}$

$= \frac{n^2(n+1)}{nt}+\frac{n+1}{nt+t-nt}$

$= \frac{n(n+1)}{t}+\frac{n+1}{t}$

$= \frac{(n+1)^2}{t}$

comments: Many students forgot to find $f''(s) > 0$ which is an essential condition for a minimum point.

- overall, very poor attempt.

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