Student Number:
Teacher:

St George Girls High School

Mathematics Extension 2

2026 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the multiple-choice answer sheet provided

  For questions in **Section II:**
  - Answer the questions in the booklets provided
  - Start each question in a new booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks : 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1– 10.
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 7 – 13)

- Attempt Questions 11 – 16.
- Allow about 2 hours and 45 minutes for this section.

Q1 - 10	/10
Q11	/15
Q12	/15
Q13	/15
Q14	/15
Q15	/15
Q16	/15
TOTAL	**/100**

Section I - 10 marks

Attempt Questions 1 – 10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. What is the value of $\frac{1-3i}{-1-3i}$?

   A. $-1+\frac{3}{5}i$

   B. -1

   C. $\frac{4}{5}+\frac{3}{5}i$

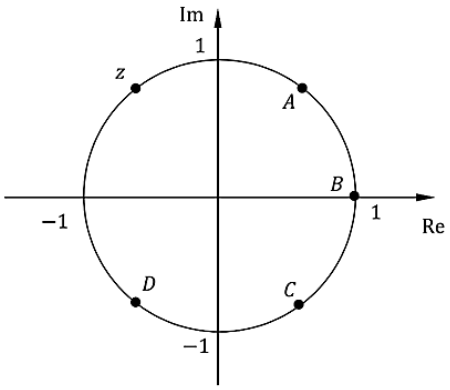
   D. $\frac{4}{5}$

2. Which of the following is the contrapositive for the statement :

   "If teachers are on holidays, then they do not answer e-mails from students."

   A. If teachers are not on holidays, then they answer e-mails from students.
   B. If teachers answer e-mails from students, then they are not on holidays.
   C. If teachers do not answer e-mails from students, then they are on holidays.
   D. If teachers are not on holidays, then they do not answer e-mails from students.

3. A complex number z lies on the unit circle in the second quadrant of the complex plane, as shown in the diagram.



   Which of the points marked A, B, C or D best shows the position of $z\bar{z}$?

   A. Point A

   B. Point B

   C. Point C

   D. Point D

4. The two lines $\underset{\sim}{r_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ intersect, where $\lambda, \mu \in \mathbb{R}$.

What is the approximate size of the angle between $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$?

A. $50°$

B. $83°$

C. $97°$

D. $140°$

5. What is the negation of the statement :

"$\forall p \in \mathbb{Z}^+, \exists q > 0$ such that $q > p$" ?

A. $\forall p \in \mathbb{Z}^+, \exists q > 0$ such that $q \leq p$.

B. $\exists p \in \mathbb{Z}^+$ such that $\forall q > 0, q \leq p$.

C. $\forall p \in \mathbb{Z}^+, \exists q > 0$ such that $q < p$.

D. $\exists p \in \mathbb{Z}^+$ such that $\forall q > 0, q < p$.

6. The line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ k \end{pmatrix}$, where $k, \lambda \in \mathbb{R}$, is a tangent to a sphere centred at $(2, 1, 0)$.

The point of contact between the tangent and the sphere is $(1, -1, 3)$.

What is the value of k?

A. $k = 2$

B. $k = 1$

C. $k = -2$

D. $k = -1$

7. Consider the even function, $f(x)$, and its primitive function, $F(x)$.

Which of the following must be equal to $\frac{1}{2}\int_0^a [f(x)+f(-x)]dx$?

A. $F\left(\frac{a}{2}\right)-F\left(-\frac{a}{2}\right)$

B. $F\left(\frac{a}{2}\right)-F(0)$

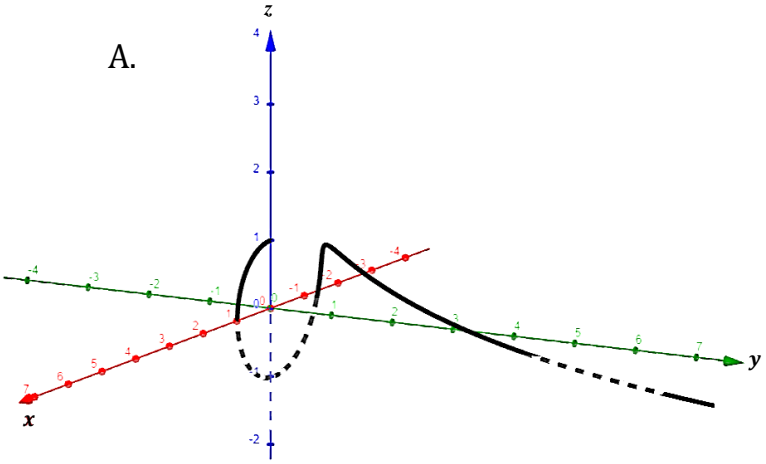
C. $F(a)-F(-a)$

D. $F(a)-F(0)$

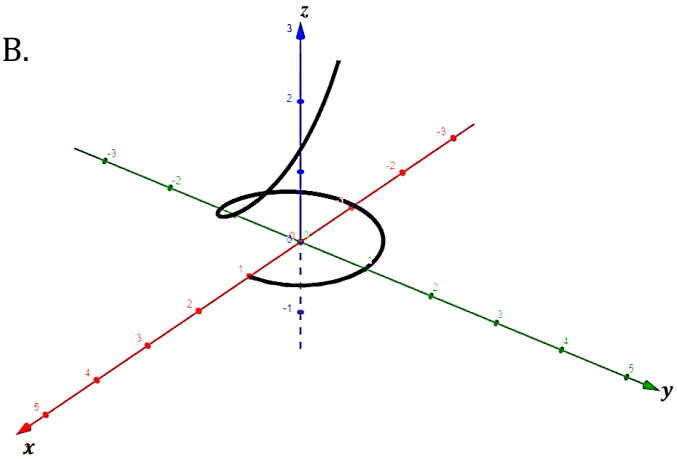
8. Which of the following diagrams best represents this parametrisation, where $t \in \mathbb{R}$?

$$\underset{\sim}{r}(t) = (\sin t)\underset{\sim}{i} + (\cos t)\underset{\sim}{j} + (e^t)\underset{\sim}{k}$$

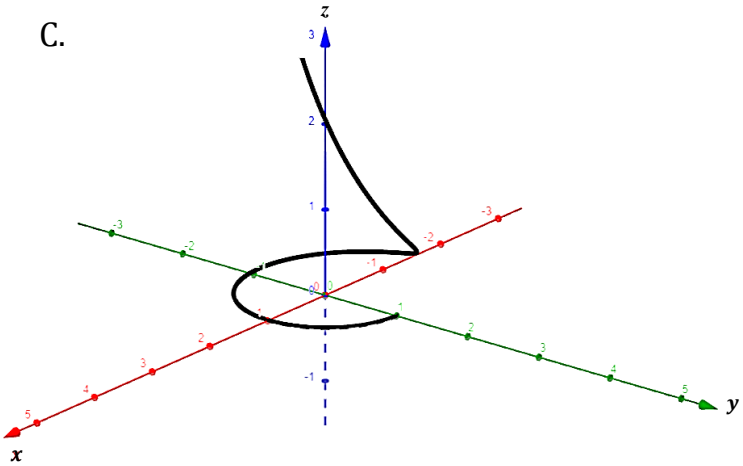
A.



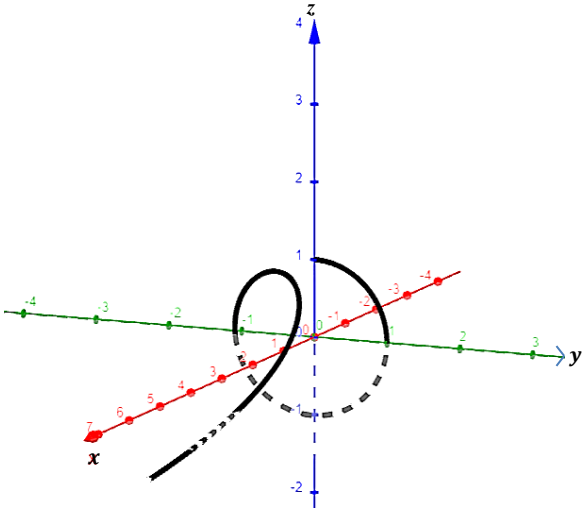
B.



C.



D.



9. Which of the following does $\int \frac{x}{\sqrt{x+5}}dx$ evaluate to?

   A. $2\sqrt{x+5}+C$

   B. $\frac{2}{3}\sqrt{(x+5)^3}+C$

   C. $\frac{2}{3}\left(\sqrt{(x+5)^3}-10\sqrt{x+5}\right)+C$

   D. $\frac{2}{3}\sqrt{(x+5)}(x-10)+C$

10. The points P and Q are represented by the complex numbers z and w respectively. If ΔOPQ is an isosceles triangle with $\angle POQ = \frac{\pi}{2}$, which of the following is not necessarily true?

    A. $z^2+w^2=0$

    B. $\frac{|z-w|}{|z|}=\sqrt{2}$

    C. $\text{Arg}(w)-\text{Arg}(z)=\frac{\pi}{2}$

    D. $|z-w|=|z+w|$

End of Section I

Section II - 90 marks

Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Marks

Question 11 (15 marks) START A NEW WRITING BOOKLET.

(a) The complex numbers w and z are given by $w = 4e^{-\frac{2i\pi}{3}}$ and $z = 2e^{\frac{i\pi}{6}}$.
By showing all necessary working, find :

(i) a simplified form for the complex number w^3. **2**

(ii) the Cartesian form for the complex number $\frac{w}{z}$. **2**

(b) Express the equation of the line $4x - y = 2$ in the form of $\underset{\sim}{r} = \underset{\sim}{u} + \lambda \underset{\sim}{v}$,
where λ is a real number. **2**

(c) Solve $z^2 - 2z + 4 - 4i = 0$. Give your answers in the form $a + ib$,
where a and b are real. **3**

(d) Using the substitution $t = \tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{3}{5\cos x + 3\sin x + 5}\, dx$. **4**

(e) Disprove the statement :

"The product of two irrational numbers is always irrational." **2**

Question 12 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) The Cartesian equation of a sphere is given as

$$x^2 - 4x + y^2 + 2y + z^2 - 6z + 5 = 0.$$

(i) Determine the centre and radius of the sphere. **2**

A line has vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

(ii) Determine the point(s) of intersection between the line and the sphere. **4**

(b) Find $\int \frac{2x-1}{x^2+2x+3}\,dx$. **3**

(c) Evaluate $\int_1^e x\ln x\,dx$. **3**

(d) Prove by contraposition that $\forall n \in N$ where $n > 1$, if $2^n - 1$ is a prime number, then n is a prime number. **3**

Question 13 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) (i) Find integers A, B and C such that:

$$\frac{x^2+3x+18}{(x^2+4)(x-5)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-5}$$ **3**

(ii) Hence, find $\int \frac{x^2+3x+18}{(x^2+4)(x-5)}\,dx$. **2**

(b) It is given that $OABC$ is a parallelogram. The points A and C are represented by position vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ respectively, and O is the origin.

(i) Find the magnitude of the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{c}$. **3**

(ii) Hence, find the area of parallelogram $OABC$. **1**

(c) Sketch the region defined by $|z-2+i| \leq 3$ and $-\frac{\pi}{3} < \arg z \leq \frac{\pi}{4}$. **3**
You are NOT required to find the coordinates of any points of intersection.

(d) Find $\int \frac{dx}{\sqrt{4x-x^2-3}}$. **3**

Question 14 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) Prove by mathematical induction that

$$\sum_{r=0}^{n}\left(\frac{1}{2}\right)^{r} = 2 - \frac{1}{2^n} \text{ for all integer } n \geq 1. \quad \textbf{3}$$

(b) It is given that ω is a non-real fifth root of unity,

(i) Prove that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **2**

(ii) Verify that $\omega - \omega^4$ is a root of $z^4 + 5z^2 + 5 = 0$. **2**

(c) Find

(i) $\int \sqrt{4 - x^2}\, dx$. **3**

(ii) $\int \frac{dx}{e^x + e^{-x}}$. **2**

(d) The vector equation of a line L_1 is represented by $\underset{\sim}{r}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$, where μ is a real number.

Let L_2 be the line passing through the point $Q(2, 4, -1)$ and perpendicular to L_1.

Find a vector representation of L_2 if it intersects L_1. **3**

Question 15 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) (i) Given that $n+1$ is a factor of $P(n)$, fully factorise

$P(n) = n^4 + 6n^3 + 11n^2 + 6n$ for $n \in Z^+$. **1**

(ii) Hence, prove by contradiction that for $n \in Z^+$,

$n^4 + 6n^3 + 11n^2 + 6n + 1$ is not divisible by 12. **3**

(b) Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$.

(i) Prove that $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **2**

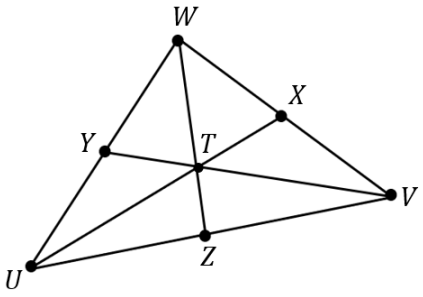
(ii) Hence, show that $2I = \int_0^{\frac{\pi}{2}} 1\, dx$. **3**

(iii) Hence, find the value of I. **1**

Question 15 continues on page 12.

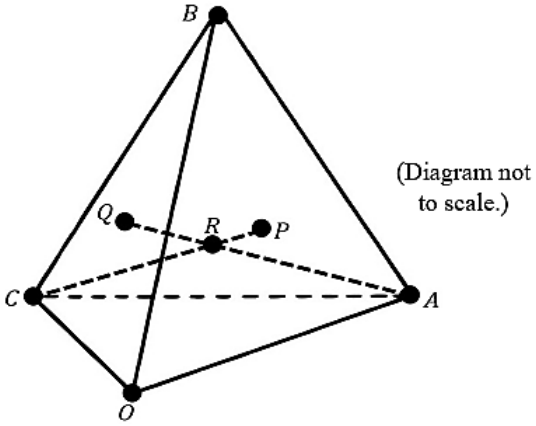
Question 15 continued **Marks**

(c) The centroid of a triangle is defined as the point where the three medians intersect. For example, in ΔUVW below, X, Y and Z are the midpoints of VW, UW and UV respectively, and T is the centroid of the triangle.



If $\overrightarrow{UV} = \underset{\sim}{v}$ and $\overrightarrow{UW} = \underset{\sim}{w}$, then the centroid T is given by $\overrightarrow{UT} = \frac{1}{3}\underset{\sim}{v} + \frac{1}{3}\underset{\sim}{w}$.
(Do NOT prove this.)

In a new figure, $OABC$ is a tetrahedron where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.
Points P and Q are the centroids of ΔOAB and ΔOBC respectively as shown in the diagram below.



Point R is the intersection of PC and QA.

(i) Given $\overrightarrow{PR} = \lambda\overrightarrow{PC}$ in ΔOPC, show that :

$$\overrightarrow{OR} = \frac{1}{3}(1-\lambda)\underset{\sim}{a} + \frac{1}{3}(1-\lambda)\underset{\sim}{b} + \lambda\underset{\sim}{c}.$$ **2**

(ii) Hence show that $\overrightarrow{OR} = \frac{1}{4}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)$. **3**

Question 16 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) Let p, q and r be three positive real numbers.

(i) Show that $\frac{p+q}{2} \geq \frac{2pq}{p+q}$. **2**

(ii) Hence, show that $\frac{1}{qr} + \frac{1}{pr} + \frac{1}{pq} \geq \frac{2}{pr+qr} + \frac{2}{pq+qr} + \frac{2}{pq+pr}$. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} x \sec x \ dx$, for $n = 0,1,2, \ldots$

You may assume for $f(x) = \sec x$, that :

$\frac{d}{dx} f(x) = \sec x \tan x$ and $\int f(x)\, dx = \ln|\sec x + \tan x| + C$.

(Do NOT prove these.)

(i) Prove that $I_n = \frac{\sqrt{2}}{2n} - \frac{2n-1}{2n} I_{n-1}$ for $n \geq 1$ **4**

(ii) Hence, or otherwise, evaluate **2**

$$\int_0^{\frac{\pi}{4}} (\sec^5 x - 2\sec^3 x + \sec x)\, dx.$$

(c) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $\sin n\theta = \frac{z^n - z^{-n}}{2i}$. **1**

(ii) You are given $\cos n\theta = \frac{z^n + z^{-n}}{2}$ (Do NOT prove this).

Hence or otherwise, prove that

$$32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\,.$$ **2**

(iii) Show that $\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{(n+1)\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$. **2**

END OF EXAMINATION

Ext 2 2026 Trial HSC Exam

MARKER'S COMMENTS

Multiple Choice

Q1 $\frac{1-3i}{-1-3i} \times \frac{-1+3i}{-1+3i}$

$= \frac{-1+3i+3i-9i^2}{1-9i^2}$

$= \frac{-1+6i+9}{1+9}$

$= \frac{8-6i}{10}$

$= \frac{4}{5} - \frac{3}{5}i$ $\therefore$ (C)

Q2 (B)

Q3 $z\bar{z} = |z|^2$ $|z|=1$

$= 1^2$

$= 1$

$\therefore$ (B)

Q4. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \sqrt{1+4+1} \times \sqrt{0+1+9} \cos\theta$

$0 + 2 - 3 = \sqrt{6} \times \sqrt{10} \cos\theta$

$\cos\theta = \frac{-1}{\sqrt{60}}$ $\therefore$ obtuse.

$\theta = 180 - \cos^{-1}\left(\frac{1}{\sqrt{60}}\right)$

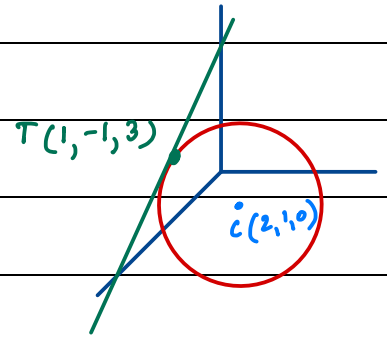
$\doteq 97^\circ$

$\therefore$ (C)

Q5. (B)

MARKER'S COMMENTS

Q6. Angle between tangent and a radii of the sphere is 90°



$$\vec{TC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ k \end{pmatrix} = 0$$

$$2 - 8 - 3k = 0$$

$$3k = -6$$

$$k = -2$$

$\therefore$ (C)

7. Even $\therefore f(x) = f(-x)$

$$\frac{1}{2}\int_0^a [f(x) + f(-x)]\,dx$$

$$= \frac{1}{2}\int_0^a f(x)\,dx + \frac{1}{2}\int_0^a f(-x)\,dx$$

$$= \frac{1}{2}\int_0^a f(x)\,dx + \frac{1}{2}\int_0^a f(x)\,dx$$

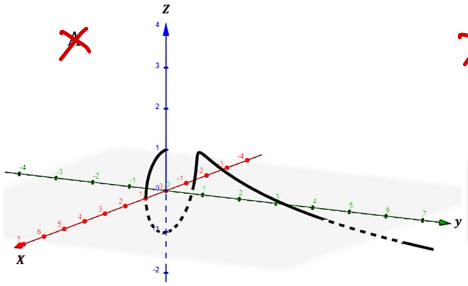
$$= \int_0^a f(x)\,dx$$

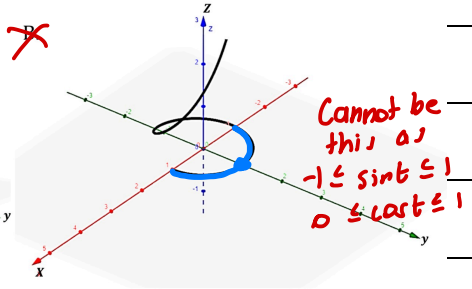
$$= [F(x)]_0^a$$

$$= F(a) - F(0) \quad \therefore$$ (D)

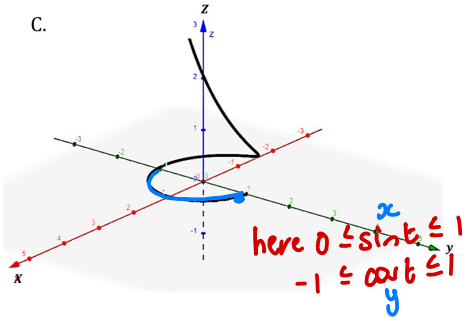
MARKER'S COMMENTS

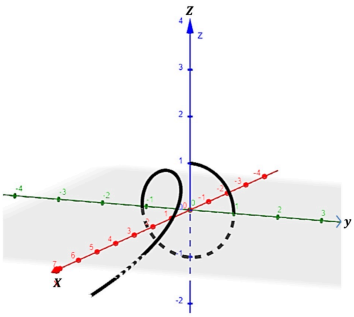
Q8. $\underset{\sim}{r}(t) = (\sin t)\underset{\sim}{i} + (\cos t)\underset{\sim}{j} + (e^t)\underset{\sim}{k}$





C.





The view in the x-y plane looks circular

$x = \sin t \qquad y = \cos t$

$x^2 = \sin^2 t \qquad y^2 = \cos^2 t$

$x^2 + y^2 = \sin^2 t + \cos^2 t$

$x^2 + y^2 = 1$ ∴ Must be B or C

$\underset{\sim}{r}(t) = (\sin t)\underset{\sim}{i} + (\cos t)\underset{\sim}{j} + e^t\underset{\sim}{k}$

$t = 0$	$(0, 1, 1)$
$t = \frac{\pi}{6}$	$(\frac{1}{2}, \frac{\sqrt{3}}{2}, e^{\pi/6})$
$t = \frac{\pi}{2}$	$(1, 0, e^{\pi/2})$
$t = \frac{5\pi}{6}$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2}, e^{5\pi/6})$
$t = \pi$	$(0, -1, e^{\pi})$

$x = \sin t$ for $0 \le t \le \pi$

$\therefore 0 \le \sin t \le 1$

$y = \cos t$ for $0 \le t \le \pi$

$-1 \le \cos t \le 1$

∴ ©

MARKER'S COMMENTS

9. $\int \frac{x}{\sqrt{x+5}}\,dx$ let $u^2 = x+5$, $x = u^2-5$, $\frac{dx}{du} = 2u$, $dx = 2u\,du$

$\int \frac{u^2-5}{\sqrt{u^2}} \times 2u\,du$

$\int \frac{u^2-5}{u} \times 2u\,du$

$2\int (u^2-5)\,du$

$= 2\left(\frac{u^3}{3} - 5u\right) + c$

$= \frac{2}{3}u^3 - 10u + c$ $\quad u^2 = x+5$, $u = \sqrt{x+5}$

$= \frac{2}{3}(x+5)\sqrt{x+5} - 10\sqrt{x+5} + c$

$= \sqrt{x+5}\left[\frac{2}{3}(x+5) - 10\right] + c$

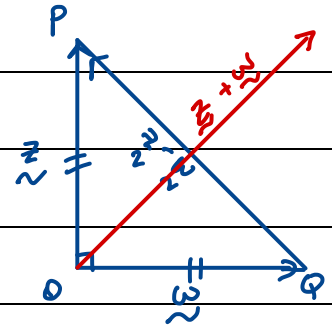
$= \frac{2}{3}\sqrt{x+5}\left[x+5 - 10\times\frac{3}{2}\right] + c$

$= \frac{2}{3}\sqrt{x+5}\,(x+5-15) + c$

(D)

MARKER'S COMMENTS

10.



Look at option A

$z = iw$ $\quad w = -iz$

$z^2 = i^2 w^2$ $\quad w^2 = i^2 z^2$

$z^2 = -w^2$ $\quad w^2 = -z^2$

$z^2 + w^2 = 0$ $\quad \therefore z^2 + w^2 = 0$

$\therefore$ A always true.

Look at option B

$|z|^2 + |w|^2 = |z - w|^2$

Isosceles

$\therefore |z| = |w|$ $\quad |z|^2 + |z|^2 = |z - w|^2$

$2|z|^2 = |z - w|^2$

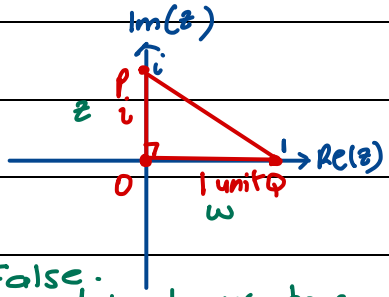
$\sqrt{2}|z| = |z - w|$

$\sqrt{2} = \dfrac{|z - w|}{|z|}$ $\quad \therefore$ B must always be true

Look at option C (by default)

$\text{Arg}(w) - \text{Arg}(z) = \dfrac{\pi}{2}$

$0 - \dfrac{\pi}{2} = -\pi/2$



(depends where you put P, Q) $\therefore$ False. Not always true

Look at option D

Diagonals of a square are equal $\therefore |z - w| = |z + w|$

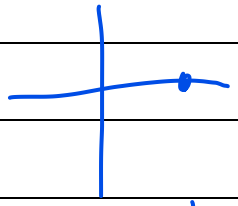
$\therefore$ (C)

MARKER'S COMMENTS

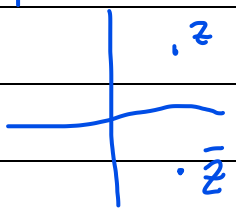
11. (a) (i) $\omega^3 = \left(4e^{-\frac{2\pi}{3}i}\right)^3$

$= 64e^{-2\pi i}$ ①

$= 64$ ①



(ii) $\frac{\omega}{z} = \frac{4e^{-\frac{2\pi}{3}i}}{2e^{-i\frac{\pi}{6}}}$



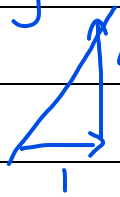
$= 2e^{-i\frac{\pi}{2}}$ ①

$= -2i$ ①

- Generally well done
- few students used z instead of $\bar{z}$
- some students had -2 instead of $-2i$ for (ii)

(b) $4x - y = 2$

$y = 4x - 2$



↑ y-int: $(0, -2)$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ where $\lambda \in \mathbb{R}$

① ①

- some students need to revisit this completely

MARKER'S COMMENTS

(c) $z^2 - 2z + (4-4i) = 0$

$z = \dfrac{2 \pm \sqrt{4 - 4(4-4i)}}{2}$

$= \dfrac{2 \pm \sqrt{-12+16i}}{2}$ (0.5)

$= 1 \pm \sqrt{-3+4i}$

Let $a + ib = -3+4i$

$\therefore a^2 - b^2 = -3$ ① } (0.5)

$2ab = 4$

$a^2 + b^2 = 5$ ②

① + ②: $2a^2 = 2$

$a^2 = 1$

$a = \pm 1$

②: $\therefore 2ab = 4$ (0.5)

$b = \dfrac{4}{2a}$

$b = \pm 2$ (0.5)

$\therefore \sqrt{-3+4i} = \pm 1 \pm 2i$

$\therefore z = 1 \pm (1+2i)$ (1)

$= 2+2i$ or $-2i$

- few students had $z = 1-1+2i$ instead of $z = 1-1-2i$
  $= 2i$ $= -2i$

MARKER'S COMMENTS

(d) $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$dx = \frac{2}{1+t^2}$ } 0.5

$x = \frac{\pi}{2} \to t = 1$

$x = 0 \to t = 0$ } 0.5

$$\therefore \int_0^{\frac{\pi}{2}} \frac{3\,dx}{5\cos x + 3\sin x + 5} = \int_0^1 \frac{3}{5\cdot\frac{1-t^2}{1+t^2} + 3\cdot\frac{2t}{1+t^2} + 5} \cdot \frac{2}{1+t^2}\,dt \quad (0.5)$$

$$= \int_0^1 \frac{6}{5 - 5t^2 + 6t + 5 + 5t^2}\,dt \quad (0.5)$$

$$= \int_0^1 \frac{6\,dt}{6t + 10} \quad (0.5)$$

$$= \left[\ln|6t+10|\right]_0^1 \quad (1)$$

$$= \ln 16 - \ln 10 \quad (0.5)$$

$$= \ln\frac{8}{5}$$

- some students remembered their t-formulae wrong.

(e) Consider $\sqrt{2}$ and $-\sqrt{2}$, both of which are irrational.

$\sqrt{2}x - \sqrt{2} = -2$ (1) for counter example

BUT -2 is rational. (1) for explanation stating rational AND irrational

$\therefore$ Statement is disproven with $\sqrt{2}$ and $-\sqrt{2}$ as counterexamples.

- Many students tried to use contrapositive and contradiction
- One counterexample is all you need to disprove a 'for all' statement

MARKER'S COMMENTS

QUESTION 12

a) $x^2 - 4x + \left(\frac{-4}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 + z^2 - 6z + \left(\frac{-6}{2}\right)^2 = -5 + \left(\frac{-4}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$

$(x-2)^2 + (y+1)^2 + (z-3)^2 = -5 + 4 + 1 + 9$

$(x-2)^2 + (y+1)^2 + (z-3)^2 = 9$

(1) - correct completion of squares and circle cartesian equation

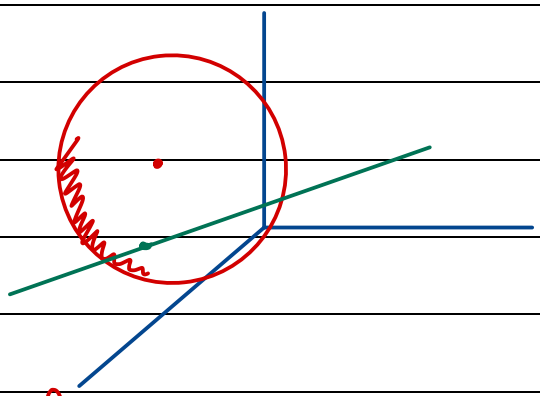
$C(2, -1, 3)$ $\quad r = \sqrt{9} = 3$

(½) - correct centre

(½) - correct radii

NOTE: No problems with this question, answered very well. **/2**

(ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$



$x = 1 + 3\lambda$

$y = -2 + 2\lambda$

$z = \lambda$

(1) writes in parametric form

$\therefore (x-2)^2 + (y+1)^2 + (z-3)^2 = 9$

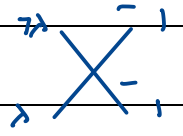
$(1 + 3\lambda - 2)^2 + (-2 + 2\lambda + 1)^2 + (\lambda - 3)^2 = 9$

(1) - correct substitution into sphere

$(3\lambda - 1)^2 + (2\lambda - 1)^2 + (\lambda - 3)^2 = 9$

$9\lambda^2 - 6\lambda + 1 + 4\lambda^2 - 4\lambda + 1 + \lambda^2 - 6\lambda + 9 = 9$

$14\lambda^2 - 16\lambda + 2 = 0$



$7\lambda^2 - 8\lambda + 1 = 0$

$(7\lambda - 1)(\lambda - 1) = 0$

$\therefore \lambda = \frac{1}{7}$ or $\lambda = 1$

(1) obtains correct values for λ.

(some ½ marks for minor algebraic errors)

MARKER'S COMMENTS

Q12 continued ...

$\lambda = 1$ $\quad x = 1 + 3\lambda$ $\quad y = -2 + 2\lambda$ $\quad z = \lambda$

$x = 1 + 3(1)$ $\quad y = -2 + 2(1)$ $\quad z = 1$

$x = 4$ $\quad y = 0$

$\therefore (4, 0, 1)$ ½ correct point of intersection

$\lambda = \frac{1}{7}$ $\quad x = 1 + 3(\frac{1}{7})$ $\quad y = -2 + 2(\frac{1}{7})$ $\quad z = \frac{1}{7}$

$x = 1\frac{3}{7}$ $\quad y = -2 + \frac{2}{7}$

$y = -\frac{12}{7}$

½ correct point of intersection

$\therefore (\frac{10}{7}, -\frac{12}{7}, \frac{1}{7})$

$\therefore$ Points of intersection are $(4, 0, 1)$ and $(\frac{10}{7}, -\frac{12}{7}, \frac{1}{7})$

/4

NOTE : This question was very well executed by the majority of students.

MARKER'S COMMENTS

Q12 continued...

b)

$$\int \frac{2x-1}{x^2+2x+3}\,dx$$

$$= \int \frac{2x+2-3}{x^2+2x+3}\,dx$$

(1) correctly manipulates the algebra

$$= \int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{x^2+2x+(\frac{2}{2})^2+3-(\frac{2}{2})^2}\,dx$$

(1) completes square on denominator

$$= \int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{(x+1)^2+3-1}\,dx$$

$$= \int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{(\sqrt{2})^2+(x+1)^2}\,dx$$

$$= \ln|x^2+2x+3| - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C$$

(½) correct integration

(½) correct integration

/3

NOTE: Most students answered this correctly. Very well done.

MARKER'S COMMENTS

Q12 continued ...

c) Let $I = \int_1^e x \ln x$ $\quad u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

$= [uv]_1^e - \int_1^e v u'$

(1) correctly applies integration by parts formula with correct values.

$= \left[\frac{x^2 \ln x}{2}\right]_1^e - \int_1^e \frac{1}{2} x \, dx$

$= \frac{e^2 \ln e}{2} - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$

(1) correctly integrates

$= \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

(1) correct evaluation

$= \frac{e^2}{2} - \frac{1}{2} \times \frac{e^2 - 1}{2}$

$= \frac{e^2}{2} - \frac{e^2 - 1}{4}$

$= \frac{e^2 + 1}{4}$

3

NOTE: Most students answered correctly, very well done.

MARKER'S COMMENTS

d) The contrapositive is:

$\forall n \in N$ when $n>1$, if n is a composite number, then 2^n-1 is a composite number.

(1) correct contrapositive statement.

Let $n = xy$ where $x, y \in N$ $x, y > 1$

(1/2) Letting n equal the product of two factors.

$2^n - 1$

$= 2^{xy} - 1$

$a^n - b^m = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{m-1})$

$= (2^x)^y - 1$

$= (2^x - 1)((2^x)^{y-1} + (2^x)^{y-2} + \ldots + (2^x)^2 + (2^x) + 1)$

$= (2^x - 1)(1 + 2^x + (2^x)^2 + \ldots + (2^x)^{y-1})$

(1) applying the factorisation technique

This is the product of two positive integers

$\therefore$ $2^n - 1$ is not prime

(1/2) correct justification

$\therefore$ By contrapositive, if $2^n - 1$ is a prime number, then n is a prime number.

/3

NOTE: Students found this question challenging. Many did not write $\forall n \in N$ where $n>1$ at the start of their contrapositive statement. When letting $n = xy$ for $\forall x, y \in N$ where $x, y > 1$ many did not write the highlighted conditions for x and y. Students did not realise they could apply the factorisation technique for the difference of terms with indices. This enabled you to obtain two integer factors for $2^n - 1$ and hence prove the contrapositive to be true.

MARKER'S COMMENTS

13. (a) (i) $x^2+3x+18 \equiv (Ax+B)(x-5)+C(x^2+4)$

At $x=5$: $25+15+18 = 29C$

$C=2$ (1)

$x=0$: $18 = -5B+4C$

$5B = 8-18$

$B=-2$ (1)

$x=1$: $1+3+18 = (A-2)(-4)+2(5)$

$22 = -4A+8+10$

$A=-1$ (1)

$\therefore A=-1, B=-2, C=2$

(ii) $\int \frac{-x-2}{x^2+4} + \frac{2}{x-5}\,dx$

(0.5)

$= \int \frac{-x}{x^2+4} - \frac{2}{x^2+4} + \frac{2}{x-5}\,dx$

$= \int \frac{-1}{2}\cdot\frac{2x}{x^2+4} - \frac{2}{x^2+4} + \frac{2}{x-5}\,dx$

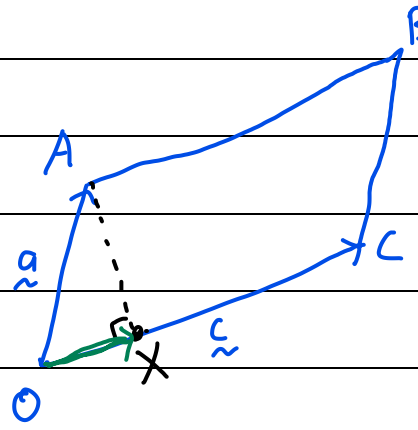
(0.5) (0.5) (0.5)

$= -\frac{1}{2}\ln(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right) + 2\ln|x-5| + C$

- Generally done very well
- Some students did not split up $\frac{-x-2}{x^2+4}$ and took this to become $-\frac{1}{2}\ln|x^2+4|$. Only 0.5 marks could be achieved due to this.

MARKER'S COMMENTS

(b) (i)



$\overrightarrow{OX} = \text{proj}_{\underset{\sim}{c}}\underset{\sim}{a}$

$= \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\begin{pmatrix}-3\\4\\2\end{pmatrix}$ (0.5) for $|\underset{\sim}{c}|$

$= \dfrac{-6+4-4}{9+16+4}\begin{pmatrix}-3\\4\\5\end{pmatrix}$

$= \dfrac{-6}{29}\begin{pmatrix}-3\\4\\5\end{pmatrix}$

$\therefore |\overrightarrow{OX}| = \dfrac{6}{29}\cdot\sqrt{29}$

$= \dfrac{6}{\sqrt{29}}$ ← (1)

(0.5) for $|\underset{\sim}{a}|$ OR $\overrightarrow{XA}$

$|\overrightarrow{XA}|^2 = |\underset{\sim}{a}|^2 - |\overrightarrow{OX}|^2$

$= (2^2+1^2+2^2) - \dfrac{36}{29}$

$= 9 - \dfrac{36}{29}$

$= \dfrac{225}{29}$ ← (1)

$\therefore |\overrightarrow{XA}| = \dfrac{15}{\sqrt{29}}$ units

(ii) Area $= b \times h$

$= |\underset{\sim}{c}| \times \dfrac{15}{\sqrt{29}}$ ← (1)

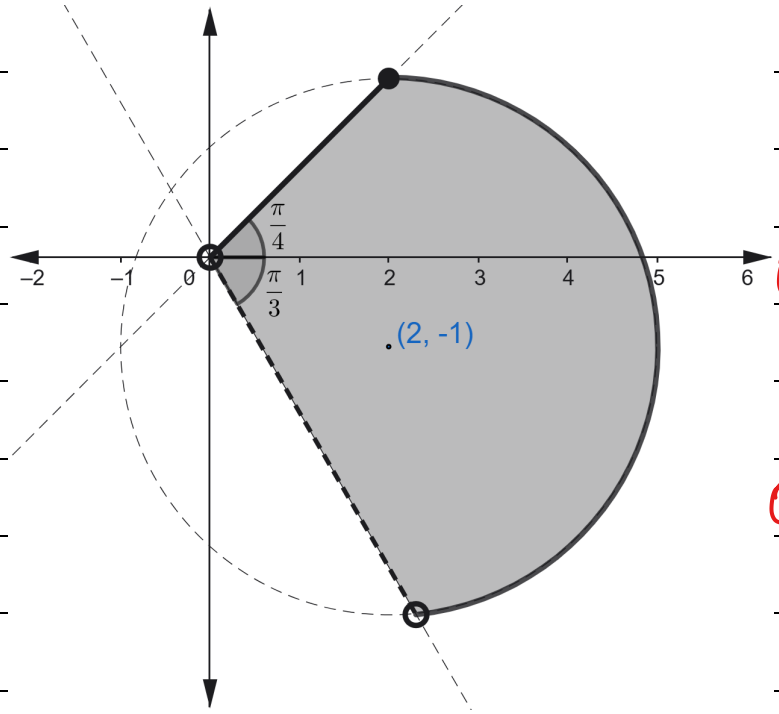
$= 15$ units^2

−0.5 if $\frac{1}{2} \times bh$

- (i) Generally well done. Watch out for incorrect transcription of your own work
- (ii) A significant proportion of students forgot how to find the area of a parallelogram

MARKER'S COMMENTS

(c)



① region of circle

① region of $-\frac{\pi}{3} < \arg(z) \leq \frac{\pi}{4}$

① detail:
- bounds of region
- open/closed circle
- indication of angles
- broken lines outside of region

- Very few students achieved full marks.
- Students need to revise inclusivity of borders and points

MARKER'S COMMENTS

(d) $\int \frac{dx}{\sqrt{4x-x^2-3}}$

$4x-x^2-3 = -x^2+4x-3$

$= -(x^2-4x)-3$

$= -(x^2-4x+4-4)-3$ ← (1)

$= -(x-2)^2+4-3$

$= 1-(x-2)^2$ ← (1)

$\therefore \int \frac{dx}{\sqrt{1-(x-2)^2}}$

$= \sin^{-1}(x-2)+C$ ← (1)

- Completed very well

MARKER'S COMMENTS

MARKER'S COMMENTS

QUESTION 14

a) Base case - Prove true for $n=1$

$$LHS = \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r \qquad RHS = 2 - \frac{1}{2^n}$$

$$= \sum_{r=0}^{1} \left(\frac{1}{2}\right)^r \qquad = 2 - \frac{1}{2^1}$$

$$= \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 \qquad = 3/2$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

(1) correctly proves base case

$\therefore$ LHS = RHS

$\therefore$ true for $n=1$

Step 2 Inductive Hypothesis – Assume true for $n=k$

$$\therefore \sum_{r=0}^{k} \left(\frac{1}{2}\right)^r = 2 - \frac{1}{2^k}$$

Step 3 Inductive step - Prove true for $n=k+1$

Prove $\sum_{r=0}^{k+1} \left(\frac{1}{2}\right)^r = 2 - \frac{1}{2^{k+1}}$

$$LHS = \sum_{r=0}^{k+1} \left(\frac{1}{2}\right)^r$$

$$= \sum_{r=0}^{k} \left(\frac{1}{2}\right)^r + \left(\frac{1}{2}\right)^{k+1}$$

$$= 2 - \frac{1}{2^k} + \left(\frac{1}{2}\right)^{k+1}$$

$$= 2 - \left(\frac{1}{2^k} - \left(\frac{1}{2}\right)^{k+1}\right)$$

$$= 2 - \left(\frac{1}{2^k} - \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)\right)$$

(½) Correct inductive hypothesis AND what to prove in inductive step.

(½) Correct split of assumption and k+1 term

(1) Correct use of assumption AND writes an argument to convincingly prove result

(-½ mark for incorrect/missing conclusion)

MARKER'S COMMENTS

Q14 continued

$= 2 - \left(\frac{1}{2}\right)\left(\frac{1}{2^k}\right)$

$= 2 - \frac{1}{2^{k+1}}$

$= RHS$ $\therefore$ true for $n = k+1$ if true for $n = k$.

Step 4 - Conclusion

$\therefore$ By the principle of mathematical induction true for all integer $n \geq 1$.

3

NOTE: Some students need to work on their understanding of sum notation and how to split them up. Otherwise, very well attempted and setting out has improved.

MARKER'S COMMENTS

Q14 continued

b) (i)

Method 1 let $z^5 = 1$

$\therefore \omega^5 = 1$ as ω is a complex root

$\therefore \omega^5 - 1 = 0$

$(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$ (1)

$\omega - 1 \neq 0$

Since $\omega \neq 1$ (given ω is non-real) (1)

$\therefore \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$

$\therefore \omega^4 + \omega^3 + \omega^2 + \omega = -1$ /2

METHOD 2 ω is a non-real fifth root of unity

$\therefore$ for $z^5 = 1$ ← real co-efficients $\therefore$ roots occur in conjugate pairs (1 mark)

$\omega^5 = 1$

$\omega^5 - 1 = 0$

$\therefore$ roots are

$1, \omega, \bar{\omega}, \omega^2, \bar{\omega}^2$

$1, \omega, \omega^2, \omega^3, \omega^4$

Sum of roots $= -\frac{b}{a}$

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = -\frac{0}{1}$ (1 mark)

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\therefore \omega + \omega^2 + \omega^3 + \omega^4 = -1$

MARKER'S COMMENTS

Q14 b) continued

(ii) $z^4 + 5z^2 + 5 = 0$

LHS $= z^4 + 5z^2 + 5$ $\quad z = \omega - \omega^4$ $\quad \omega^4 + \omega^3 + \omega^2 + \omega = -1$, $\omega^5 = 1$

$= (\omega - \omega^4)^4 + 5(\omega - \omega^4)^2 + 5$ (1)

$= \omega^4 - 4\omega^3(\omega^4) + 6\omega^2(\omega^4)^2 - 4\omega(\omega^4)^3 + (\omega^4)^4 + 5(\omega^2 - 2\omega^5 + \omega^8) + 5$

$= \omega^4 - 4\omega^7 + 6\omega^{10} - 4\omega^{13} + \omega^{16} + 5\omega^2 - 10\omega^5 + 5\omega^8 + 5$

$= \omega^4 - 4\omega^2 + 6 - 4\omega^3 + \omega + 5\omega^2 - 10 + 5\omega^3 + 5$ from a) $\omega^5 = 1$

$= \omega^4 + \omega^2 + 1 + \omega^3 + \omega$

$= 0$ from (i)

(1/2) correct simplification to prove result using (i)

(1/2) uses $\omega^5 = 1$ multiple times to simplify expression.

$\therefore$ $z = \omega - \omega^4$ is a solution of $z^4 + 5z^2 + 5 = 0$ /2

NOTE: Many errors occured with the algebra in (ii). Some students substituted and then did not expand to prove the result. Students who used the roots and their conjugate could not prove the result convincingly with exact answers rather than decimal approximations.

MARKER'S COMMENTS

Q14 continued ...

c) (i) $\int \sqrt{4-x^2}\, dx$

Let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$= \int \sqrt{4-(2\sin\theta)^2} \times 2\cos\theta\, d\theta$

$= \int \sqrt{4-4\sin^2\theta} \times 2\cos\theta\, d\theta$

(1) mark to correctly substitute in terms of θ

$= \int \sqrt{4(1-\sin^2\theta)} \times 2\cos\theta\, d\theta$

$= 2\int \sqrt{\cos^2\theta} \times 2\cos\theta\, d\theta$

$= 2\int 2\cos^2\theta\, d\theta$

$= \frac{4}{2}\int (1+\cos 2\theta)\, d\theta$

$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

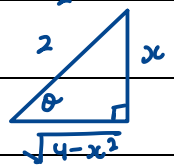
(1) mark correctly finds the primitive

$= 2\left[\theta + \frac{1}{2} \times 2\sin\theta\cos\theta\right] + C$

$= 2\left[\sin^{-1}\frac{x}{2} + \frac{x}{2} \times \frac{\sqrt{4-x^2}}{2}\right] + C$

Now $x = 2\sin\theta$

$\frac{x}{2} = \sin\theta \rightarrow \theta = \sin^{-1}\frac{x}{2}$



and $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left[\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{4}\right] + C$

(½) (½) correctly finds the primitive in terms of x with use of triangle.

3

NOTE: Some students forgot that the answer needs to be in terms of x.

MARKER'S COMMENTS

Q14 c) continued ...

(ii) $\int \frac{dx}{e^x + e^{-x}} \quad \times e^x \quad \times e^x$

(1) multiplies numerator and denominator by e^x

$= \int \frac{e^x}{e^{2x} + 1} dx$

$= \int \frac{e^x}{1 + (e^x)^2} dx$

(1) correct answer

(if "u" sub was used, answer needed to be in terms of x otherwise ½ mark deducted)

$= \tan^{-1} e^x + C$

/2

OR

$\int \frac{dx}{e^x + e^{-x}} \quad \times e^{-x} \quad \times e^{-x}$

$= \int \frac{e^{-x}}{1 + e^{-2x}} dx$

$= -\int \frac{-e^{-x}}{1 + (e^{-x})^2} dx$

$= -\tan e^{-x} + C$

NOTE: Quite well done with many different methods. Some students incorrectly thought it was a ln answer.

MARKER'S COMMENTS

Q14 continued ...

d) method 1

Let $\underset{\sim}{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 2 \\ 3-\mu \end{pmatrix}$ Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the point P of intersection

$Q(2,4,-1)$ $P\begin{pmatrix} 1+\mu \\ 2 \\ 3-\mu \end{pmatrix}$

$\therefore \overrightarrow{QP} = \begin{pmatrix} 1+\mu-2 \\ 2-4 \\ 3-\mu+1 \end{pmatrix}$

$= \begin{pmatrix} \mu-1 \\ -2 \\ 4-\mu \end{pmatrix}$

We want $\overrightarrow{QP} \perp L_1$

$\therefore \begin{pmatrix} \mu-1 \\ -2 \\ 4-\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$

$\therefore \mu - 1 + 0 - 4 + \mu = 0$

$2\mu = 5$

$\mu = 5/2$ (1) obtains μ by use of the dot product

$\therefore \overrightarrow{QP} = \begin{pmatrix} 5/2 - 1 \\ -2 \\ 4 - 5/2 \end{pmatrix}$

$= \begin{pmatrix} 3/2 \\ -2 \\ 3/2 \end{pmatrix}$ (1) correct $\overrightarrow{QP}$

$\therefore L_2 : \underset{\sim}{r}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3/2 \\ -2 \\ 3/2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$ (1) Correct equation for L_2

/3

MARKER'S COMMENTS

Q14 d) continued ...

METHOD 2 $L_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$ let $L_2 = \begin{pmatrix}2\\4\\-1\end{pmatrix} + \lambda\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\therefore \begin{pmatrix}1\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

$x - z = 0$

$\therefore x = z$

$\therefore L_1 = \begin{pmatrix}1+\mu\\2\\3-\mu\end{pmatrix}$ $L_2 = \begin{pmatrix}2+\lambda x\\4+\lambda y\\-1+\lambda z\end{pmatrix}$

(1) correctly states $x = z$ and equates L_1 and L_2

Intersection Point

$\begin{pmatrix}1+\mu\\2\\3-\mu\end{pmatrix} = \begin{pmatrix}2+\lambda x\\4+\lambda y\\-1+\lambda x\end{pmatrix}$

$1+\mu = 2+\lambda x \cdots (1)$ $2 = 4+\lambda y \cdots (2)$ $3-\mu = -1+\lambda x \cdots (4)$

$\lambda x = 3 - \mu + 1$

$\lambda x = 4 - \mu \cdots (5)$

Sub (5) into (1)

$1+\mu = 2+4-\mu$

$2\mu = 5$

$\mu = \frac{5}{2}$

Intersection is $\begin{pmatrix}1+\mu\\2\\3-\mu\end{pmatrix}$

$= \begin{pmatrix}1+5/2\\2\\3-5/2\end{pmatrix}$

$= \begin{pmatrix}7/2\\2\\1/2\end{pmatrix}$

(1) correctly finds $\mu = \frac{5}{2}$ and uses it to find the point of intersection

MARKER'S COMMENTS

Let the intersection be $P\begin{pmatrix}7/2\\2\\1/2\end{pmatrix}$ Now $Q\begin{pmatrix}2\\4\\-1\end{pmatrix}$

$$\therefore \quad \vec{QP} = \begin{pmatrix}7/2-2\\2-4\\1/2+1\end{pmatrix}$$

$$= \begin{pmatrix}3/2\\-2\\3/2\end{pmatrix}$$

$\therefore$ L_2 is represented by $\underset{\sim}{r}_2$

$$\underset{\sim}{r}_2 = \begin{pmatrix}2\\4\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3/2\\-2\\3/2\end{pmatrix}$$

(1) correctly finds $\vec{QP}$ and uses it to find the equation of L_2

NOTE: Students found this question challenging. Many different methods were used. Revise vector equations of lines in detail.

MARKER'S COMMENTS

QUESTION 15

a) (i) $P(n) = n^4 + 6n^3 + 11n^2 + 6n$

$= n(n^3 + 6n^2 + 11n + 6)$

$= n(n+1)(n^2 + 5n + 6)$

$= n(n+1)(n+2)(n+3)$

$\therefore (n+1)$ is a factor of $n^3 + 6n^2 + 11n + 6$

$$n+1 \overline{)\, n^3 + 6n^2 + 11n + 6}$$

$n^3 + n^2$ (−)

$5n^2 + 11n + 6$

$5n^2 + 5n$ (−)

$6n + 6$

$6n + 6$

0

/1

(ii) Assume $n^4 + 6n^3 + 11n^2 + 6n + 1$ is divisible by 12 (1) correct assumption

ie $n^4 + 6n^3 + 11n^2 + 6n + 1 = 12M \quad M \in Z$

Now, from (i) $P(n) = n^4 + 6n^3 + 11n^2 + 6n$ (1) correct reasoning for why $P(n) \div$ by 12

Now From (i) $= n(n+1)(n+2)(n+3)$ Any set of four consecutive numbers

$= 12P \quad P \in Z^+$ for $n \in Z^+$ is

We assumed $n^4 + 6n^3 + 11n^2 + 6n + 1 = 12M$ divisible by 24

$\therefore 12P + 1 = 12M$ $\therefore$ also divisible by 12

however this cannot be true (1) correct reasoning as to why the assumption is false and correct setting out of proof.

as $12P + 1$ is not divisible by 12.

$\therefore$ the negation is false

$\therefore$ By contradiction the original statement must be true

$\therefore n^4 + 6n^3 + 11n^2 + 6n + 1$ is not divisible by 12

/3

MARKER'S COMMENTS

Q15 a)(ii) continued ...

NOTE: Some students could not factorise the expression in (i) using polynomials or substitution or use of calculator method.

For (ii) there needed to be a clear explanation for the multiplication of four consecutive integers being a multiple of 24 and hence a multiple of 12.

Please review solution on how to set the rest of the proof out mathematically rather than using long worded explanations.

MARKER'S COMMENTS

Q15 continued

c) (i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$RHS = \int_0^a f(a-x)\,dx$

Let $u = a - x$

$\frac{du}{dx} = -1$

$du = -dx \quad \therefore dx = -du$

When $x = a$, $u = a - a = 0$

$x = 0$, $u = a - 0 = a$

$= \int_a^0 f(u)(-du)$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$u \to x$ by dummy variable

$= \int_0^a f(x)\,dx$

$= LHS.$

(1) mark correct u-substitution, derivative and bounds

some ½ marks for no mention of bounds

(1) correctly sets out proof to obtain result.

NOTE: Well done by those who learnt this proof from their theory notes. You needed to have some mention of considering bounds to gain full mark.

Please SHOW how you got them and avoid "squiggle" notation.

Eg $u = a - x$ NOT $a \sim 0$

$x = a$, $u = a - a = 0$

MARKER'S COMMENTS

Q15 continued ...

c) (ii) METHOD 1

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \quad \ldots (1)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx$$

(1) mark uses (i)

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \quad \ldots (2)$$

(1) mark uses $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ etc ...

Add (1) + (2)

$$I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$\therefore 2I = \int_0^{\pi/2} 1\, dx$$

(1) show the addition of the two "I" result and correctly proves result.

/3

NOTE: Some students did not revise this concept in detail.

Many students were successfull with varying methods.

MARKER'S COMMENTS

METHOD 2

$LHS = 2I$

$$= 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \quad \ldots (1)$$

$$= 2\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

(1) mark uses (i)

$$= 2\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \quad \ldots (2)$$

(1) mark uses $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ etc ...

$$2I = 2\int_0^{\pi/2} \left(\frac{\cos x + \sin x}{\cos x + \sin x} - \frac{\sin x}{\cos x + \sin x}\right) dx$$

$$= 2\int_0^{\pi/2} 1\,dx - 2\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx$$

(1) add and take away $\sin x$ to obtain integral of 1 and I and correctly proves result.

$$2I = 2\int_0^{\pi/2} 1\,dx - 2I$$

$$4I = 2\int_0^{\pi/2} 1\,dx$$

$$\therefore 2I = \int_0^{\pi/2} 1\,dx$$

MARKER'S COMMENTS

$x = \frac{\pi}{2} - u$

c) (i) $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

Let $u = \frac{\pi}{2} - x$

$\frac{du}{dx} = -1$

$dx = -du$

$x = \pi/2 \rightarrow u = \frac{\pi}{2} - \frac{\pi}{2} = 0$

$x = 0 \rightarrow u = \frac{\pi}{2}$

$I = \int_{\pi/2}^{0} \frac{\sin\left(\frac{\pi}{2} - u\right)}{\sin\left(\frac{\pi}{2} - u\right) + \cos\left(\frac{\pi}{2} - u\right)} (-du)$ (1)

$I = \int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u}\, du$ (1)

$I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u}\, du$

$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$

By the Dummy Variable

(1)

$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$

$\therefore 2I = \int_0^{\pi/2} 1\, dx$

(iii) $2I = [x]_0^{\pi/2}$

$2I = \frac{\pi}{2} - 0$ (1) correct answer.

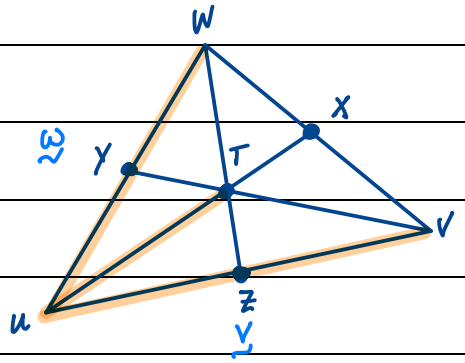
$I = \pi/4$

1

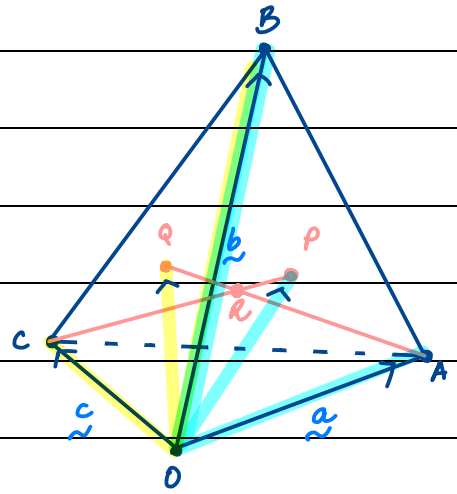
MARKER'S COMMENTS

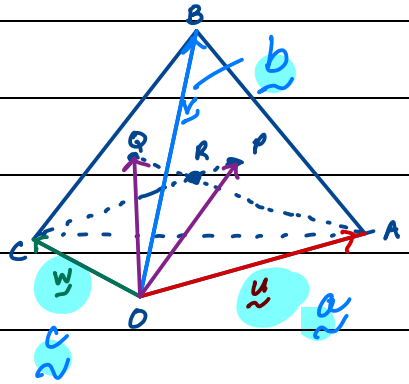
Q15 continued . . .

c)



Given $\overrightarrow{UT} = \frac{1}{3}\underset{\sim}{v} + \frac{1}{3}\underset{\sim}{w}$





Given $\overrightarrow{PR} = \lambda\overrightarrow{PC}$, $\overrightarrow{OP} = \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$

and $\overrightarrow{OQ} = \frac{1}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{c}$

$\overrightarrow{OQ} = \frac{1}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{c}$ and $\overrightarrow{OP} = \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$

(½) correctly obtains $\overrightarrow{OQ}$ or $\overrightarrow{OP}$ using the given information

$\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC}$

$\overrightarrow{PC} = \overrightarrow{OC} - \overrightarrow{OP}$

$= \underset{\sim}{c} - \left(\frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}\right)$

$= \underset{\sim}{c} - \frac{1}{3}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b}$

(½) correctly finds $\overrightarrow{PC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \overrightarrow{OP} + \lambda\overrightarrow{PC}$

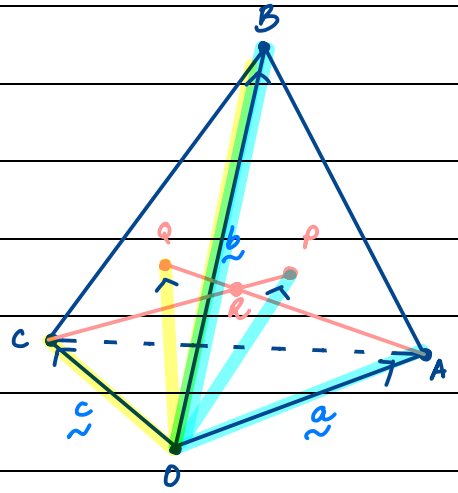
$= \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b} + \lambda\left(\underset{\sim}{c} - \frac{1}{3}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b}\right)$

$= \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b} + \lambda\underset{\sim}{c} - \frac{1}{3}\lambda\underset{\sim}{a} - \frac{1}{3}\lambda\underset{\sim}{b}$

$= \frac{1}{3}(1-\lambda)\underset{\sim}{a} + \frac{1}{3}(1-\lambda)\underset{\sim}{b} + \lambda\underset{\sim}{c}$

(1) correctly shows how to obtain $\overrightarrow{OR}$

MARKER'S COMMENTS



$\vec{OA} = \vec{OQ} + \vec{QA}$

$\vec{QA} = \vec{OA} - \vec{OQ}$

$= \underset{\sim}{a} - (\frac{1}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{c})$

$\vec{OR} = \vec{OQ} + \vec{QR}$

$= \vec{OQ} + \mu\vec{QA}$

$= \frac{1}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{c} + \mu(\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c})$

$= \frac{1}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{c} + \mu\underset{\sim}{a} - \frac{1}{3}\mu\underset{\sim}{b} - \frac{1}{3}\mu\underset{\sim}{c}$

$= \frac{1}{3}(1-\mu)\underset{\sim}{b} + \frac{1}{3}(1-\mu)\underset{\sim}{c} + \mu\underset{\sim}{a}$

(1) correctly finds another expression for $\vec{OR}$

$\therefore \vec{OR} = \frac{1}{3}(1-\lambda)\underset{\sim}{a} + \frac{1}{3}(1-\lambda)\underset{\sim}{b} + \lambda\underset{\sim}{c}$ $\quad \vec{OR} = \frac{1}{3}(1-\mu)\underset{\sim}{b} + \frac{1}{3}(1-\mu)\underset{\sim}{c} + \mu\underset{\sim}{a}$

$\therefore \frac{1}{3}(1-\lambda)\underset{\sim}{a} + \frac{1}{3}(1-\lambda)\underset{\sim}{b} + \lambda\underset{\sim}{c} = \frac{1}{3}(1-\mu)\underset{\sim}{b} + \frac{1}{3}(1-\mu)\underset{\sim}{c} + \mu\underset{\sim}{a}$

$\therefore \frac{1}{3}(1-\lambda) = \mu \qquad \frac{1}{3}(1-\lambda) = \frac{1}{3}(1-\mu) \qquad \lambda = \frac{1}{3}(1-\mu)$.

$1-\lambda = 3\mu \qquad 1-\lambda = 1-\mu$

$\lambda = 1-3\mu \cdots (1) \qquad \lambda = \mu \cdots (2)$

(1) Equates the two expressions for $\vec{OR}$

Sub 2 into (1)

$\mu = 1-3\mu$

$4\mu = 1$

$\mu = \frac{1}{4}$ and $\lambda = \frac{1}{4}$

(1) correctly finds μ or $\lambda = 1/4$ using the two expressions for $\vec{OR}$ (not by working backwards from the result)

$\therefore \vec{OR} = \frac{1}{3}(1-\frac{1}{4})\underset{\sim}{a} + \frac{1}{3}(1-\frac{1}{4})\underset{\sim}{b} + \frac{1}{4}\underset{\sim}{c}$

$= \frac{1}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b} + \frac{1}{4}\underset{\sim}{c}$

$= \frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$

MARKER'S COMMENTS

NOTE: Students found this question (particularly (ii)) very challenging. More experience for 3D vector proofs is advised.

16. (a)(i)

Method 1:

$$(p-q)^2 \geq 0$$
$$p^2 - 2pq + q^2 \geq 0$$
$$p^2 + 2pq + q^2 \geq 4pq$$
$$(p+q)^2 \geq 4pq$$

divide both sides by $2(p+q)$

$$\therefore \quad \frac{p+q}{2} \geq \frac{2pq}{p+q} \quad \text{since } p, q > 0$$

Method 2:

$$\text{LHS} - \text{RHS} = \frac{p+q}{2} - \frac{2pq}{p+q}$$
$$= \frac{(p+q)^2 - 4pq}{2(p+q)}$$
$$= \frac{p^2 + 2pq + q^2 - 4pq}{2(p+q)}$$
$$= \frac{p^2 - 2pq + q^2}{2(p+q)}$$
$$= \frac{(p-q)^2}{2(p+q)}$$
$$\geq 0 \quad \text{since } p, q > 0 \text{ and } (p-q)^2 \geq 0$$

$$\therefore \quad \text{LHS} \geq \text{RHS as required}$$

Marking Criteria:

- One mark was awarded for reaching the line just before completing the square
- The final mark was awarded for getting to the correct solution

General Comments

- generally done well
- some students that used method 1 ended up with

$$p^2 + q^2 \geq 2pq 2p^2 + 2q^2 \geq 4pq$$

- Look at what you're trying to prove to get a better idea on what to do. $p + q$ informs you that obtaining $(p+q)^2$ is a better option.

16. (a)(ii)

Method 1

From (i), $\frac{p+q}{2} \geq \frac{2pq}{p+q}$ ①

Similarly, $\frac{p+r}{2} \geq \frac{2pr}{p+r}$ ② and $\frac{q+r}{2} \geq \frac{2qr}{q+r}$ ③

① + ② + ③ : $\frac{p+q}{2} + \frac{p+r}{2} + \frac{q+r}{2} \geq \frac{2pq}{p+q} + \frac{2pr}{p+r} + \frac{2qr}{q+r}$

$\therefore \quad p+q+r \geq \frac{2pq}{p+q} + \frac{2pr}{p+r} + \frac{2qr}{q+r}$

$\frac{1}{pqr}(p+q+r) \geq \frac{1}{pqr}\left(\frac{2pq}{p+q} + \frac{2pr}{p+r} + \frac{2qr}{q+r}\right)$

$\therefore \quad \frac{1}{qr} + \frac{1}{pr} + \frac{1}{pq} \geq \frac{2}{p+q} + \frac{2}{p+r} + \frac{2}{q+r}$

Method 2:

From (i), $\frac{p+q}{2} \geq \frac{2pq}{p+q}$ $\therefore$ $\frac{pr+qr}{2} \geq \frac{2pqr^2}{pr+qr}$

$\frac{2}{pr+qr} \leq \frac{p+q}{2pqr}$

$\frac{2}{pr+qr} \leq \frac{1}{2}\left(\frac{1}{qr} + \frac{1}{pr}\right)$

$\frac{1}{2}\left(\frac{1}{qr} + \frac{1}{pr}\right) \geq \frac{2}{pr+qr}$ ①

Similarly, $\frac{1}{2}\left(\frac{1}{pq} + \frac{1}{qr}\right) \geq \frac{2}{pq+qr}$ ②

$\frac{1}{2}\left(\frac{1}{pq} + \frac{1}{pr}\right) \geq \frac{2}{pq+pr}$ ③

① + ② + ③ :

$\frac{1}{2}\left(\frac{1}{qr} + \frac{1}{pr}\right) + \frac{1}{2}\left(\frac{1}{pq} + \frac{1}{qr}\right) + \frac{1}{2}\left(\frac{1}{pq} + \frac{1}{pr}\right) \geq \frac{2}{pr+qr} + \frac{2}{pq+qr} + \frac{2}{pq+pr}$

$\frac{1}{pr} + \frac{1}{qr} + \frac{1}{pq} \geq \frac{2}{pr+qr} + \frac{2}{pq+qr} + \frac{2}{pq+pr}$

Method 3:

$$\text{From (i), } \frac{p+q}{2} \geq \frac{2pq}{p+q} \quad \therefore \quad \frac{\frac{1}{pr}+\frac{1}{qr}}{2} \geq \frac{2\frac{1}{pr}\frac{1}{qr}}{\frac{1}{pr}+\frac{1}{qr}}$$

$$\frac{qr+pr}{2pqr^2} \geq \frac{2}{qr+pr}$$

$$\frac{1}{2pr}+\frac{1}{2qr} \geq \frac{2}{qr+pr} \quad ①$$

$$\text{Similarly, } \frac{1}{2pq}+\frac{1}{2qr} \geq \frac{2}{qr+pq} \quad ②$$

$$\frac{1}{2pr}+\frac{1}{2pq} \geq \frac{2}{pr+pq} \quad ③$$

① + ② + ③ :

$$\frac{1}{2pr}+\frac{1}{2qr}+\frac{1}{2pq}+\frac{1}{2qr}+\frac{1}{2pr}+\frac{1}{2pq} \geq \frac{2}{qr+pr}+\frac{2}{qr+pq}+\frac{2}{pr+pq}$$

$$\frac{1}{pr}+\frac{1}{qr}+\frac{1}{pq} \geq \frac{2}{pr+qr}+\frac{2}{pq+qr}+\frac{2}{pq+pr}$$

Marking Criteria:

- One mark was awarded for generating equations that could lead to a correct solution
- The final mark was awarded for getting to the correct solution
- half a mark was deducted for minor errors, or being one step away from the END of the solution

General Comment:

- many students that started with method 2 did not reciprocate the inequality until AFTER adding all three solutions together. This is ILLEGAL.

16. (b)(i)

$$I_n = \int_0^{\pi/4} \tan^{2n-1} x \tan x \sec x \, dx \qquad u = \tan^{2n-1} x \qquad v' = \tan x \sec x$$
$$u' = (2n-1)(\tan^{2n-1} x)(\sec^2 x) \qquad v = \sec x$$

$$= \left[\tan^{2n-1} x \sec x\right]_0^{\pi/4} - \int_0^{\pi/4} (2n-1) \tan^{2n-2} x \sec^3 x \, dx$$

$$= \left[(\tan \tfrac{\pi}{4})^{2n-1} \sec \tfrac{\pi}{4} - 0\right] - (2n-1) \int_0^{\pi/4} \tan^{2n-2} x \sec x \, (1 + \tan^2 x) \, dx$$

$$= \left(1^{2n-1} \times \sqrt{2}\right) - (2n-1) \int_0^{\pi/4} \left(\tan^{2n-2} x \sec x + \tan^{2n} x \sec x\right) dx$$

$$= \sqrt{2} - (2n-1) \int_0^{\pi/4} \tan^{2n-2} x \sec x \, dx - (2n-1) \int_0^{\pi/4} \tan^{2n} x \sec x \, dx$$

$$I_n = \sqrt{2} - (2n-1) \, I_{n-1} - (2n-1) \, I_n$$

$$I_n + (2n-1) \, I_n = \sqrt{2} - (2n-1) \, I_{n-1}$$

$$I_n \, (1 + 2n - 1) = \sqrt{2} - (2n-1) \, I_{n-1}$$

$$I_n = \frac{\sqrt{2}}{2n} - \frac{2n-1}{n} \, I_{n-1}$$

Marking Criteria:

- one mark for the correct split
- one mark for correct application of IBP
- one mark for appropriately writing expressions in terms of I_n and I_{n-1}
- one mark for the final result
- half mark deducted for minor errors such as writing $2n-1$ instead of $2n-2$ in simplification process

General comment:

- A significant proportion of students did not split correctly.
  LOOK at the NUMERATOR of the coefficient of I_{n-1} as a clue for how to split it. Because the numerator was $2n-1$, we should split $\tan^{2n} x$ into $\tan x \tan^{2n-1} x$
- Many students differentiated $\tan^{2n-1} x$ incorrectly. Don't forget to apply the chain rule and multiply by $\sec^2 x$.
- Some students had difficulty recognising I_{n-1} as needing a power of $2n-2$, and instead took it as needing $2n-1$.
- A few students took $\tan^2 x = 1 - \sec^2 x$

16. (b)(ii)

$$\begin{aligned}\int_0^{\pi/4}\left(\sec^5 x - 2\sec^3 x + \sec x\right)dx &= \int_0^{\pi/4}\sec x\left(\sec^4 x - 2\sec^2 x + 1\right)\\ &= \int_0^{\pi/4}\sec x\left(\sec^2 x - 1\right)^2\\ &= \int_0^{\pi/4}\sec x\tan^4 x\\ &= \int_0^{\pi/4}\tan^4 x\sec x\\ &= I_2\end{aligned}$$

$$\begin{aligned}I_0 &= \int_0^{\pi/4}\tan^0 x\sec x\,dx\\ &= \int_0^{\pi/4}\sec x\,dx\\ &= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}\\ &= \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|\\ &= \ln\left|\sqrt{2}+1\right| - \ln|1+0|\\ &= \ln\left(\sqrt{2}+1\right)\end{aligned}$$

$$I_1 = \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right)$$

$$\begin{aligned}I_2 &= \frac{\sqrt{2}}{4} - \frac{3}{4}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right)\right)\\ &= \frac{3}{8}\ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{8}\end{aligned}$$

Marking Criteria:

- one mark reaching or understanding the integral is equivalent to I_2
- one mark for determining I_2
- IF none of the above were achieved, half a mark was awarded for substituting $\sec^2 x$ with $1+\tan^2 x$ and some correct manipulation

General comment:

- A few students saw the power of 4 and took it as I_4.

16(c)(i)

$$z = \cos\theta + i\sin\theta$$

$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

$$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

$$\text{RHS} = \frac{z^n - z^{-n}}{2i}$$
$$= \frac{\cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)}{2i}$$
$$= \frac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{2i}$$
$$= \frac{2i\sin n\theta}{2i}$$
$$= \sin n\theta$$
$$= \text{LHS}$$

Marking Criteria:

- One mark awarded for correct solution
- As it is a show question, we want to see the subtraction, not simply $cis(n\theta) - cis(-n\theta) = 2i\sin\theta$. This was deducted half a mark.

General comment:

- Generally done well

16(c)(ii)

$$
\begin{aligned}
32\sin^4\theta\cos^2\theta &= 32(\sin\theta)^4(\cos\theta)^2 \\
&= 32\left(\frac{z^1 - z^{-1}}{2}\right)^4\left(\frac{z^1 + z^{-1}}{2}\right)^2 \\
&= \frac{32\left(z^4 - 4z^3z^{-1} + 6z^2z^{-2} - 4z^nz^{-3n} + z^{-4}\right)}{16} \times \frac{\left(z^2 + 2z^1z^{-1} + z^{-2}\right)}{4} \\
&= \frac{1}{2}\left(z^4 + z^{-4} - 4z^2 - 4z^{-2} + 6\right)\left(z^2 + z^{-2} + 2\right) \\
&= \frac{1}{2}\left(z^6 + z^{-6} - 2z^4 - 2z^{-4} - z^2 - z^{-2} + 4\right) \\
&= \frac{z^6 + z^{-6}}{2} - 2\cdot\frac{z^4 + z^{-4}}{2} - \frac{z^2 + z^{-2}}{2} + 2 \\
&= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2
\end{aligned}
$$

Marking Criteria:

- One mark for substituting $\frac{z - z^{-1}}{2i} = \sin\theta$ and $\frac{z + z^{-1}}{2} = \cos\theta$
- One mark for full proof
- If none of the above marks were gained, 0.5 marks awarded if you found the trig expression for $\cos 6\theta$ via binomial expansion. This approach will lead to the correct solution, but will take an extremely long time to achieve.

General comment:

- Very poorly attempted, but understandable.
- Due to the wording 'or otherwise', you can approach this as a difficult Mathematics Extension 1 trig question instead

16(c)(iii)

Let $z = e^{i\theta}$, so $\cos(k\theta) = \dfrac{z^k + z^{-k}}{2}$. Then:

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{1}{2}\left(\sum_{k=1}^{n} z^k + \sum_{k=1}^{n} z^{-k}\right)$$

$$\sum_{k=1}^{n} z^k = \frac{z^{n+1} - z}{z-1}, \qquad \sum_{k=1}^{n} z^{-k} = \frac{1 - z^{-n}}{z-1}$$

$$\therefore \sum_{k=1}^{n} z^k + \sum_{k=1}^{n} z^{-k} = \frac{z^{n+1} - z + 1 - z^{-n}}{z-1}$$

$$\therefore \sum_{k=1}^{n} \cos k\theta = \frac{1}{2} \cdot \frac{z^{n+1} - z + 1 - z^{-n}}{z-1}$$

$$= \frac{1}{2} \cdot \frac{z^{-1/2}(z^{n+1} - z + 1 - z^{-n})}{z^{-1/2}(z-1)}$$

$$= \frac{1}{2} \cdot \frac{z^{n+(1/2)} - z^{-n-(1/2)} - z^{1/2} + z^{-1/2}}{z^{1/2} - z^{-1/2}}$$

$$= \frac{1}{2} \cdot \frac{2i\sin\left(\dfrac{(2n+1)\theta}{2}\right) - 2i\sin\left(\dfrac{\theta}{2}\right)}{2i\sin\left(\dfrac{\theta}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\sin\left(n + \frac{\theta}{2}\right) - \sin\left(\dfrac{\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)}$$

Using $\sin P - \sin Q = 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$ with $P = n + \dfrac{\theta}{2}, Q = \dfrac{\theta}{2}$:

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{1}{2} \cdot \frac{2\cos\left(\dfrac{(n+1)\theta}{2}\right)\sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)}$$

$$= \frac{\cos\left(\dfrac{(n+1)\theta}{2}\right)\sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)}$$

LHS $= \cos\theta + \cos 2\theta + \cdots + \cos n\theta$, which equals $\mathrm{Re}(S_n)$ below.

Let $S_n = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{in\theta}$

$$\therefore \quad a = e^{i\theta}, \quad r = e^{i\theta}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{e^{i\theta}\left(1-(e^{i\theta})^n\right)}{1-e^{i\theta}}$$

$$= \frac{e^{i\theta}\cdot e^{in\theta/2}\left(e^{-in\theta/2}-e^{in\theta/2}\right)}{e^{i\theta/2}\cdot\left(e^{-i\theta/2}-e^{i\theta/2}\right)}$$

$$= \frac{e^{i\theta}\left[e^{in\theta/2}\left(-2i\sin\frac{n\theta}{2}\right)\right]}{e^{i\theta/2}\left(-2i\sin\frac{\theta}{2}\right)}$$

$$= \frac{e^{i\theta/2}\left(e^{in\theta/2}\sin\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$= \frac{e^{i\frac{(n+1)\theta}{2}}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\therefore \mathrm{Re}(S_n) = \frac{\cos\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \text{ since } e^{\frac{(n+1)\theta i}{2}} = \cos\frac{n+1}{2}\theta + i\sin\frac{n+1}{2}\theta$$

Marking Criteria:

- One mark for the correct approach with the correct sum formula
- Final mark for the correct solution

General comment:

- very poorly attempted, but understandable
- The wording of the question did not imply you had to use previous results. You can approach this through induction or other methods.