SYDNEY BOYS HIGH SCHOOL

Name:

NESA Number:

Tick next to your teacher's name

AF		**PSP**	
JJ		**JT**	
		AYW	

2026

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Only NESA approved calculators may be used
- A reference sheet is provided with this paper
- All answers, unless otherwise stated, should be left in simplified, exact form
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations as answers that rely totally on calculator technology may not necessarily receive full marks.

Total Marks: 100

Section I – 10 marks (pages 2 – 8)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 9 – 16)

- Attempt all Questions in Section II
- Allow about 2 hour and 45 minutes for this section

Examiner:
External Examiner

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

Question 1

Which is the correct expression for $\int \dfrac{3x+1}{x^2+2x+4}dx$?

A. $\dfrac{3}{2}\ln(x^2+2x+4) - \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right) + C$

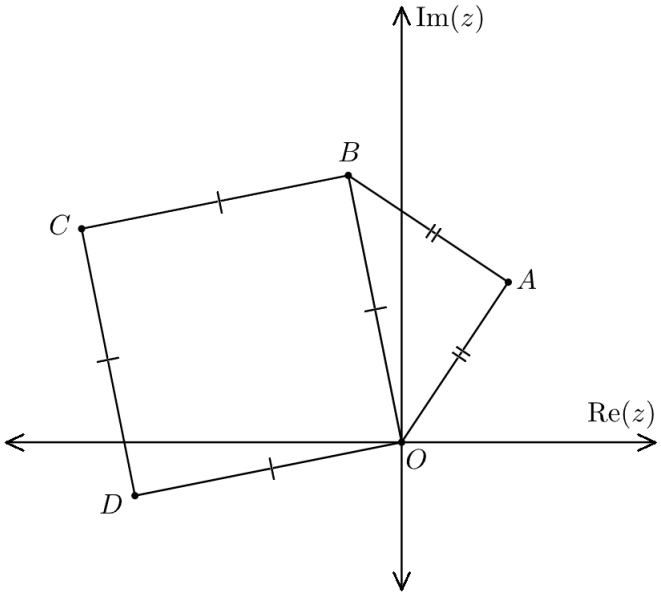
B. $3\ln(x^2+2x+4) - \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right) + C$

C. $\dfrac{3}{2}\ln(x^2+2x+4) - 2\tan^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right) + C$

D. $3\ln(x^2+2x+4) - 2\tan^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right) + C$

Question 2

In the diagram, $\triangle OAB$ is a right-angled isosceles triangle and $OBCD$ is a square. Let A represent the complex number α and D represent the complex number β.



Which of the following is correct?

A. $\beta + \alpha = i\alpha$

B. $\beta - \alpha = i\alpha$

C. $\sqrt{2}\beta + \alpha = i\alpha$

D. $\sqrt{2}\beta - \alpha = i\alpha$

Question 3

A particle moving in simple harmonic motion centred at the origin attains minimum acceleration of -3 m/s^2 and a maximum speed of 4 m/s. At what position does it attain three quarters of its maximum acceleration?

A. $x = \dfrac{16}{3}$

B. $x = -\dfrac{16}{3}$

C. $x = 4$

D. $x = -4$

Question 4

Let u be a complex number with $|u| \leq 5$. What is the minimum value of $|2u + 3 + 4i|$?

A. -5

B. 0

C. 10

D. 15

Question 5

Let $\text{Arg}\,(z)$ be the principal argument of z. How many possible real values of k can satisfy

$$\text{Arg}\,(zw) = \text{Arg}\,(z) + \text{Arg}\,(w) + k\pi$$

for complex numbers z and w?

A. 2

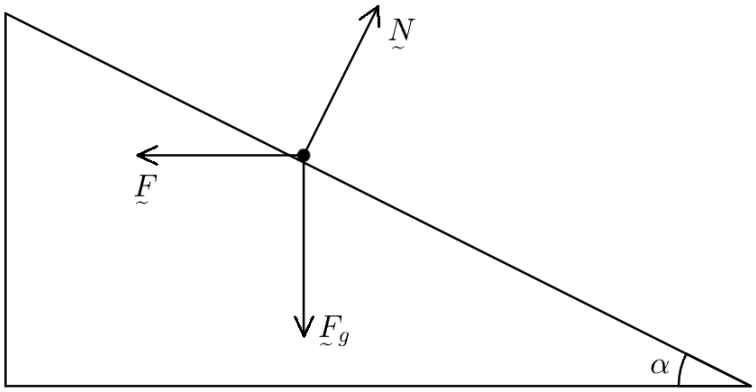
B. 3

C. 4

D. 5

Question 6

A ball is held stationary on a slope by a horizontal force $\underset{\sim}{F}$ and is subject to gravity $\underset{\sim}{F}_g$ and the normal force $\underset{\sim}{N}$ as shown.



Which of the following is equal to $\underset{\sim}{F} \cdot \underset{\sim}{N}$?

A. $|\underset{\sim}{F}||\underset{\sim}{F}_g|$

B. $|\underset{\sim}{F}|^2$

C. $-|\underset{\sim}{F}||\underset{\sim}{F}_g|$

D. $-|\underset{\sim}{F}|^2$

Question 7

Let $P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_0$ be a polynomial with integer coefficients, where n is an even positive integer.

$P(z)$ is known to have zeros at $z = 2 + \sqrt{3}$, $z = -\sqrt{3}$ and $z = -4 + 2i$.

What is the minimum possible degree of the $P(z)$?

A. 6

B. 8

C. 10

D. 12

Question 8

f is an even continuous function. Which of the following is not always equal to $\int_0^{2a} f(x)dx$?

A. $\int_0^a \Big[f(x) + f(2a - x) \Big] dx$

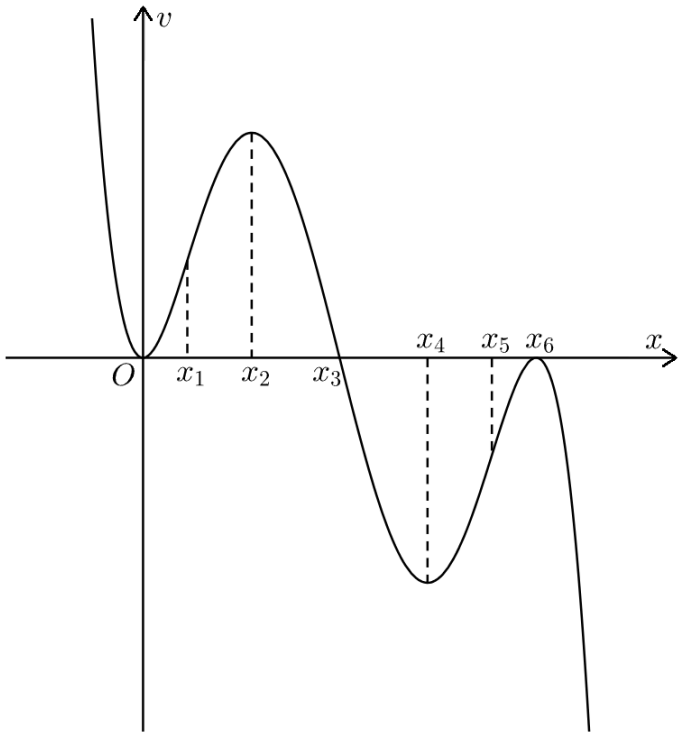
B. $\int_{-a}^0 \Big[f(x) + f(a - x) \Big] dx$

C. $\int_{-a}^0 \Big[f(x) + f(-2a - x) \Big] dx$

D. All of the above are equal.

Question 9

A particle is initially placed at a random position on the x-axis. The particle's velocity v as a function of x is shown in the diagram below.



At what range of initial positions will it experience positive acceleration?

A. $0 < x < x_2$ or $x_4 < x < x_6$

B. $x < x_1$ or $x_3 < x < x_5$

C. $0 < x < x_2$ or $x_3 < x < x_4$ or $x > x_6$

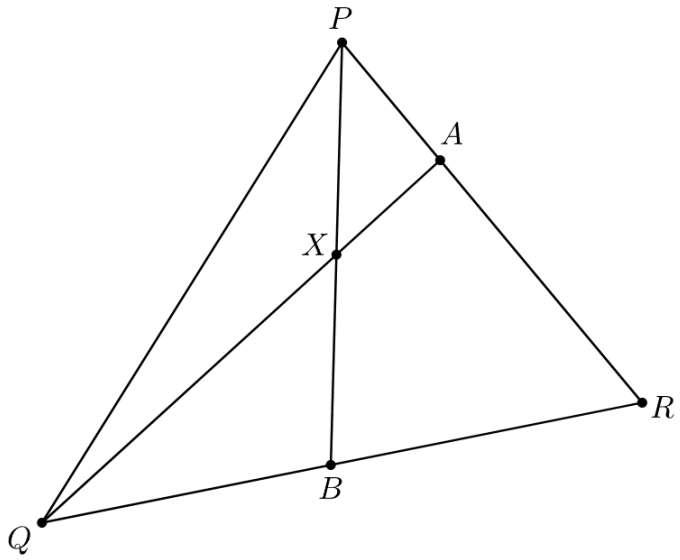
D. $0 < x < x_2$ or $x_3 < x < x_6$

Question 10

The diagram shows triangle $\triangle PQR$.

A is on PR such that $PA : PR = 1 : 4$.

B is on QR such that $QB : BR = m : n$ where m, n are odd positive integers.



If the point W divides an interval UV so that $UW : WV = a : b$ and the position vectors of U and V are $\underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively, then $\overrightarrow{OW} = \dfrac{a}{a+b}\underset{\sim}{v} + \dfrac{b}{a+b}\underset{\sim}{u}$.

Let X be on AQ such that $AX : XQ = x : y$ where x, y are positive integers that add to 47. By finding two different expressions for $\overrightarrow{PX}$, what is the minimum value of x?

A. 2

B. 3

C. 4

D. 5

Section II

Attempt Questions 11 – 16
Allow 2 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet.
Extra writing booklets are available.
In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $r = -1 + 9i$ and $s = 6 + 3i$. Evaluate

(i) $2r + \overline{s}$; **1**

(ii) $\left|\dfrac{3}{rs}\right|$; **1**

(iii) $\text{Im}\left(\dfrac{s}{r}\right)$. **2**

(b) (i) Find the values of constants A, B and C such that **2**

$$\frac{10}{(t+2)(t^2+1)} = \frac{A}{t+2} + \frac{Bt+C}{t^2+1}.$$

(ii) By using a suitable substitution, deduce for some positive integers p and q, that **4**

$$\int_0^{\frac{\pi}{2}} \frac{5\sin x}{1+2\sin x - \cos x}dx = \pi + \ln\left(\frac{p}{q}\right).$$

State the values of p and q.

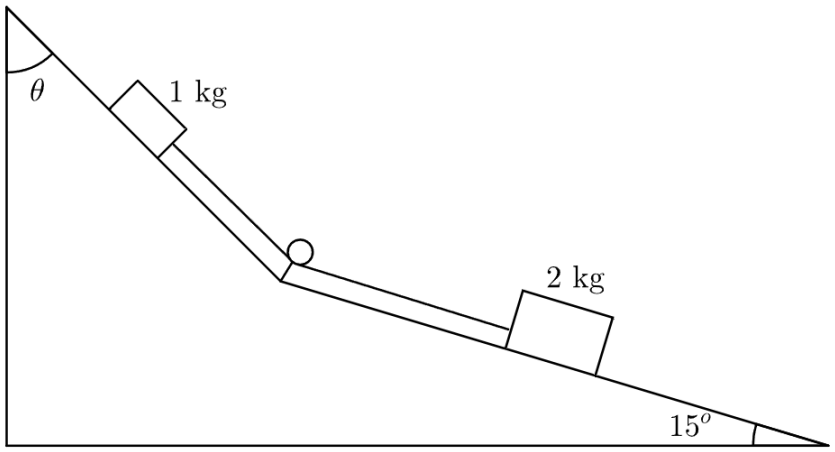
(c) Sketch the values of z for which $|z-2i| \leq 1$ and $0 \leq \arg(z+1-2i) < \dfrac{\pi}{4}$ **3**
are all satisfied simultaneously.

(d) Points A, B and C are on the Cartesian plane such that $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$. **2**

By use of a diagram, or otherwise, find the size of angle $\angle BCA$ to the nearest minute.

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Two blocks, of mass 2 kg and 1 kg are initially stationary on a frictionless ramp as shown. A light, inextensible rope joins the two masses and is hooked under a frictionless pulley. The masses are then released.



(i) Assuming the rope is taut, find the acceleration of the 1 kg mass, leaving your answer in terms of g and θ, where θ is an acute angle. **2**

(ii) Find the range of values of θ such that the rope remains taut. **2**

(b) (i) Find $\int (xe^x + e^x)\, dx$. **1**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{x+1}{\sqrt{e^{-2x} - x^2}} dx$. **2**

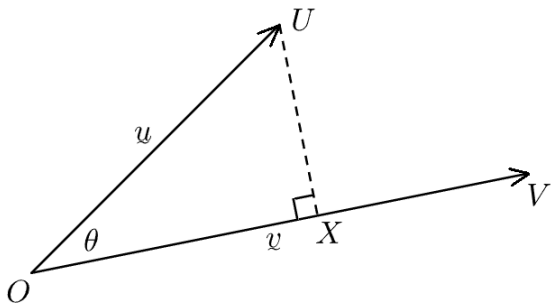
Question 12 continues on page 11

(c) A particle moves horizontally in a line on the x-axis. It initially is moving at a speed of u m/s away from the origin such that $u > 1$. It experiences a pushing force numerically equal to $\dfrac{mk}{v}$ N due to the ground. There is also drag numerically equal to mkv N acting on the particle.

(i) Show that $\dfrac{d^2x}{dt^2} = k\left(\dfrac{1}{v} - v\right)$. **1**

(ii) Prove that the time taken for the particle to stop is $T = \dfrac{1}{2k}\ln(u^2 - 1)$. **3**

(d) Let U and V have position vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively.



(i) Show that $\overrightarrow{XU} = \underset{\sim}{u} - \text{proj}_{\underset{\sim}{v}}\,\underset{\sim}{u}$. **1**

(ii) Hence, or otherwise, prove that $||\underset{\sim}{v}|\underset{\sim}{u} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\hat{\underset{\sim}{v}}| = |\underset{\sim}{u}||\underset{\sim}{v}|\sin\theta$ where θ is the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

End of Question 12

Question 13 (16 marks) Use a SEPARATE writing booklet.

(a) For all positive x, a function f satisfies

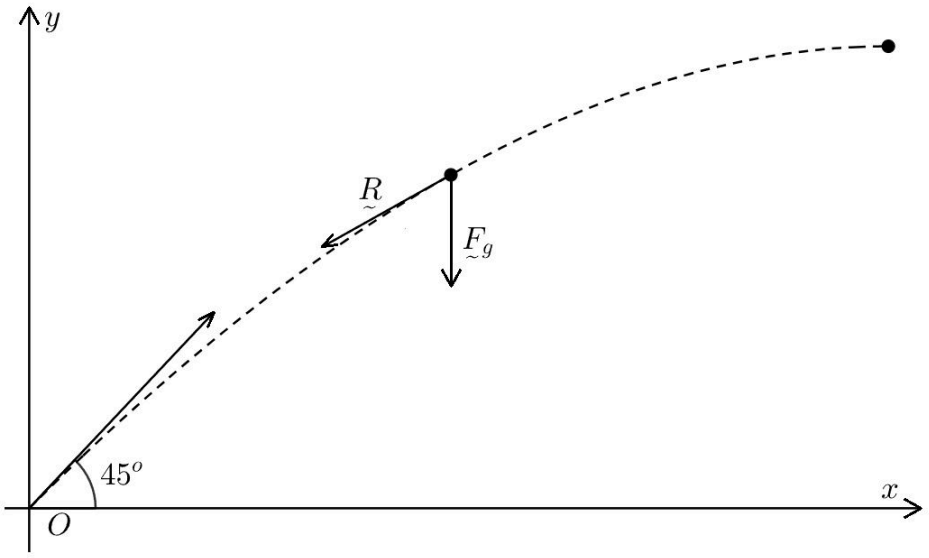
$$2f(x) - f\left(\frac{1}{x}\right) = \frac{1}{x} + x \ln x.$$

(i) Explain why $2f\left(\frac{1}{x}\right) - f(x) = x - \frac{\ln x}{x}$ **1**

(ii) Evaluate $\int_1^2 f(x)\, dx$. **4**

Question 13 continues on page 13

(b) A 5 g flare is shot at 45^o with speed $10\sqrt{2}$ from the origin and is programmed to detonate when it reaches its maximum height.



There is air resistance numerically proportional to the velocity squared, $\underset{\sim}{R} = -k\begin{pmatrix}\dot{x}^2\\ \dot{y}^2\end{pmatrix}$, acting on the flare, where $\dot{x}$ and $\dot{y}$ are the horizontal and vertical velocities of the flare respectively. Let $k = 2.5 \times 10^{-4}$ and $g = 9.8$ m/s^2.

(i) Show that the acceleration vector $\underset{\sim}{a}$ of the particle is $\underset{\sim}{a} = \begin{pmatrix}-0.05\dot{x}^2\\ -9.8 - 0.05\dot{y}^2\end{pmatrix}$. **1**

(ii) Prove that $x = 20\ln\left(1 + \dfrac{t}{2}\right)$. **3**

(iii) Prove that during the upwards motion, $\dot{y} = \dfrac{14(5 - 7\tan(0.7t))}{7 + 5\tan(0.7t)}$. **2**

(iv) Hence, find the time until detonation, to 2 decimal places. **1**

(v) Prove that the vertical displacement of the flare is **3**

$$y = 20\ln\left|\cos(0.7t) + \frac{5}{7}\sin(0.7t)\right|.$$

(vi) Hence, find the position at which the flare will detonate, to 2 decimal places. **1**

End of Question 13

Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \displaystyle\int_a^1 x^2(\ln x)^n \, dx$, where $a > 0$ and n is a non-negative integer.

(i) Prove that $I_n = -\dfrac{a^3(\ln a)^n}{3} - \dfrac{n}{3} I_{n-1}$, and state the values of n for which this is true. **4**

(ii) Define $J_n = \lim\limits_{a \to 0} I_n$. **4**

Hence, deduce that $J_n = \dfrac{(-1)^n n!}{3^{n+1}}$ for $n \geq 0$.

(b) A mass m kg is shot upwards at initial speed u m/s in a medium that applies a resistive force numerically equal to mkv N, where $k > 0$ is the constant of proportionality and v is the velocity of the mass in m/s. The acceleration due to gravity is g m/s^2. Take upwards as positive. Prove that the object's displacement satisfies $v - u + gt + kx = 0$. **4**

(c) The probability density function $\phi(z)$ of a standard normal variable $Z \sim N(0, 1)$ is given by $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, for $-\infty < z < \infty$.

You may assume that $E(Z^n) = \displaystyle\int_{-\infty}^{\infty} z^n \phi(z) \, dz = \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} z^n \phi(z) \, dz$.

(i) Explain why $E(Z^2) = 1$. **1**

(ii) Hence, or otherwise, prove that $E(Z^4) = 3$. **3**

Question 15 (14 marks) Use a SEPARATE writing booklet.

(a) Let z, w be complex numbers that satisfy **4**

$$|z - 5 - 3i| = 1 \qquad \text{and} \qquad |w - 2 - 4i| = 1.$$

Find the range of values of $\text{Arg}\,(z - w)$, where Arg denotes the principal argument.

(b) Let n points $P_1, P_2, \ldots, P_n$, which are represented by the position vectors $\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots, \underset{\sim}{p}_n$ respectively, lie on a straight line l. **4**

Prove by mathematical induction, for $n \geq 2$, that the point represented by

$$\underset{\sim}{u} = \lambda_1 \underset{\sim}{p}_1 + \lambda_2 \underset{\sim}{p}_2 + \cdots + \lambda_n \underset{\sim}{p}_n$$

is collinear with the n points, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.

(c) (i) Let a, b and c be real numbers. Prove that if $a \sin x + b \cos x = c$ has no solutions for all real x, then $a^2 + b^2 < c^2$. **2**

(ii) Particle P is released from rest 4 m to the right of the origin, and follows motion governed by the equation $\ddot{x}_1 = 8 - 4x_1$. **4**

Particle Q undergoes simple harmonic motion centred at the origin. Q is released T seconds later from the origin, with the same period and amplitude as particle P and negative velocity.

For what range of values of T during particle P's first oscillation should Q be released so that the two particles never meet?

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\beta = \dfrac{-3+\sqrt{5}}{2}$.

(i) Show that $\beta^2 + 3\beta + 1 = 0$, and hence, prove that $\beta^{-2} + 3\beta^{-1} + 1 = 0$. **2**

(ii) Let $B_n = \beta^n + \beta^{-n}$. Deduce that $B_n = -3B_{n-1} - B_{n-2}$. **1**

(iii) Hence, briefly explain why $B_n \in \mathbb{Z}$ for all positive integers n. **3**

(b) Consider the polynomial $P(z) = z^{2n+1} - 1$.

(i) Prove that the zeros of $P(z)$ are $z = \alpha_k = \cos\theta_k + i\sin\theta_k$ where $\theta_k = \dfrac{2k\pi}{2n+1}$ for $k = 0, 1, 2, \ldots, 2n$. **1**

(ii) Prove that $\overline{\alpha_k} = \alpha_{2n+1-k}$. **1**

(iii) It is known that **2**

$$\prod_{k=1}^{n} T_k = T_1 \times T_2 \times T_3 \times \cdots \times T_{n-1} \times T_n.$$

Hence, or otherwise, prove that

$$z^{2n} + z^{2n-1} + \cdots + z + 1 = \prod_{k=1}^{n} (z - \alpha_k)(z - \overline{\alpha_k}).$$

(iv) Let $A_k = 5 - 4\sin^2\left(\dfrac{k\pi}{2n+1}\right)$. **2**

Show that $A_k = (\alpha_k - \beta)\left(1 - \alpha_k^{-1}\beta^{-1}\right)$, where β is the value from (a).

(v) Hence, or otherwise, prove that $S = \displaystyle\prod_{k=1}^{n}\left[5 - 4\sin^2\left(\frac{k\pi}{2n+1}\right)\right]$ is an integer. **3**

You may assume the result from Question 16 (a) (iii).

End of Paper

SYDNEY BOYS HIGH SCHOOL

2026

YEAR 12

HSC TASK 4

Mathematics Extension 2

Sample Solutions

NOTE: This process of checking your mark is about reading the solutions and the comments. Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	2	3	4	5	6	7	8	9	10
A	A	D	B	B	D	C	D	C	B

(a) Let $r = -1 + 9i$ and $s = 6 + 3i$. Evaluate

(i) $2r + \bar{s}$; **1**

$= 2(-1+9i) + (\overline{6+3i})$

$= -2 + 18i + 6 - 3i$

$= 4 + 15i$

(ii) $\left|\frac{3}{rs}\right|$; **1**

$= \frac{|3|}{|r||s|}$

$= \frac{3}{\sqrt{(-1)^2+(9)^2}\sqrt{(6)^2+(3)^2}}$

$= \frac{3}{\sqrt{82}\sqrt{45}}$

$= \frac{3}{\sqrt{82}\,3\sqrt{45}} = \frac{1}{\sqrt{410}}$

(iii) $\text{Im}\left(\frac{s}{r}\right)$. **2**

$= \text{Im}\left(\frac{6+3i}{-1+9i} \times \frac{-1-9i}{-1-9i}\right)$

$= \text{Im}\left(\frac{-6-54i-3i+27}{i+81}\right)$

$= \text{Im}\left(\frac{21-57i}{82}\right)$

$= -\frac{57}{82}$

Comment

This was answered well by the majority of students.

(b) (i) Find the values of constants A, B and C such that **2**

$$\frac{10}{(t+2)(t^2+1)} = \frac{A}{t+2} + \frac{Bt+C}{t^2+1}.$$

$10 = A(t^2+1) + (Bt+C)(t+2)$

let $t=-2$

$10 = A((-2)^2+1)$

$10 = 5A$

$A = 2$

equate coeff of t^2

$0 = 2 + B$

$B = -2$

equate constants

$10 = 2 + 2C$

$2C = 8$

$C = 4$

$\therefore A = 2, B = -2, C = 4$

Comment

this was done well by the majority of students

(b) (continued)

(ii) By using a suitable substitution, deduce for some positive integers p and q, that **4**

$$\int_0^{\frac{\pi}{2}} \frac{5\sin x}{1+2\sin x-\cos x}dx = \pi + \ln\left(\frac{p}{q}\right).$$

State the values of p and q.

$$I = \int_0^{\frac{\pi}{2}} \frac{5\sin x}{1+2\sin x-\cos x}\,dx$$

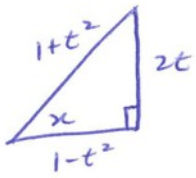
let $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = 2 \cdot \frac{1}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$



when $x=0$, $t=0$; $x=\frac{\pi}{2}$, $t=1$

$$I = \int_0^1 \frac{5\cdot\frac{2t}{1+t^2}}{1+2\cdot\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{20t\,dt}{(1+t^2)^2+4t(1+t^2)-(1-t^2)(1+t^2)}$$

$$= \int_0^1 \frac{20t\,dt}{(1+t^2)(1+t^2+4t-(1-t^2))}$$

$$= \int_0^1 \frac{20t\,dt}{(1+t^2)(2t^2+4t)}$$

$$= \int_0^1 \frac{20t\,dt}{(1+t^2)2t(t+2)}$$

(b) (ii) (continued)

$$I = \int_0^1 \frac{10\,dt}{(1+t^2)(t+2)}$$

$$= \int_0^1 \left(\frac{2}{t+2} + \frac{-2t+4}{t^2+1} \right) dt$$

$$= \int_0^1 \left(2 \cdot \frac{1}{t+2} - \frac{2t}{t^2+1} + 4 \cdot \frac{1}{t^2+1} \right) dt$$

$$= \left[2\ln|t+2| - \ln|t^2+1| + 4\tan^{-1} t \right]_0^1$$

$$= 2\ln|1+2| - \ln|1^2+1| + 4\tan^{-1}1 - \left(2\ln|0+2| - \ln|0^2+1| + 4\tan^{-1}0 \right)$$

$$= 2\ln 3 - \ln 2 + 4 \cdot \frac{\pi}{4} - 2\ln 2$$

$$= \pi + 2\ln 3 - 3\ln 2$$

$$= \pi + \ln 3^2 - \ln 2^3$$

$$= \pi + \ln 9 - \ln 8$$

$$= \pi + \ln\left(\frac{9}{8}\right)$$

$\therefore p = 9, q = 8.$

Comment

Students should be expecting to use part (i). Given sine and cosine functions in denominator, t-results should be used.

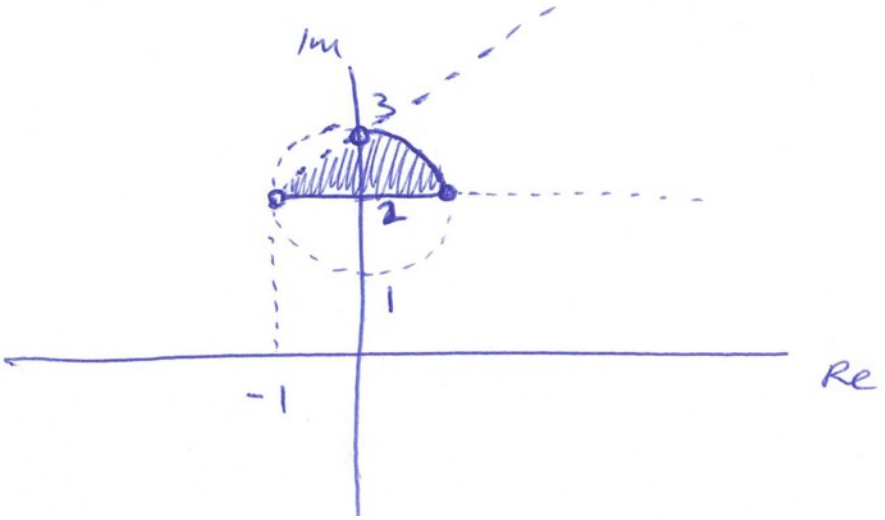
(c) Sketch the values of z for which $|z-2i| \leq 1$ and $0 \leq \arg(z+1-2i) < \frac{\pi}{4}$ are all satisfied simultaneously. **3**

$|z-2i| \leq 1$

"distance from (0,2) is less than or equal to 1"

$0 \leq \arg(z+1-2i) < \frac{\pi}{4}$

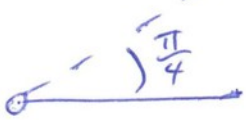
"argument of z from (-1,2) is between 0 & $\frac{\pi}{4}$"



Comments

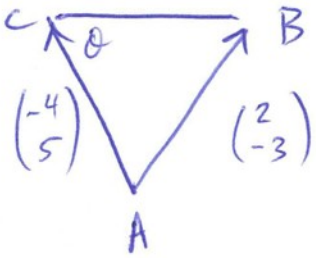
It is good technique to dash all lines and curves to begin with.
They can be changed to unbroken lines later.
Open circles should be used at (-1,2) and (0,3)

An argument of $\frac{\pi}{4}$ from (-1,2) will pass through (0,3).
If this is not shown the angle should be shown on the diagram. ie.



(d) Points A, B and C are on the Cartesian plane such that $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$. **2**

By use of a diagram, or otherwise, find the size of angle $\angle BCA$ to the nearest minute.



In order to find $\angle BCA$ we need to use $\overrightarrow{CB}$ & $\overrightarrow{CA}$

$$\overrightarrow{CA} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$$

$$= \begin{pmatrix} 4 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -8 \end{pmatrix}$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (4)(6) + (-5)(-8)$$

$$= 64$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = |\overrightarrow{CA}||\overrightarrow{CB}| \cos\theta$$

$$= \sqrt{(4)^2 + (-5)^2}\sqrt{(6)^2 + (-8)^2}\cos\theta$$

$$= 10\sqrt{41}\cos\theta$$

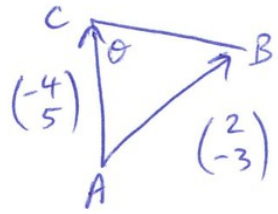
$$10\sqrt{41}\cos\theta = 64$$

$$\cos\theta = \frac{64}{10\sqrt{41}}$$

$$\theta = \cos^{-1}\left(\frac{64}{10\sqrt{41}}\right) \approx 1°47'$$

(d) (continued)

Alternatively,



$|\vec{AB}| = \sqrt{(2)^2 + (-3)^2}$

$= \sqrt{13}$

$|\vec{AC}| = \sqrt{(-4)^2 + (5)^2}$

$= \sqrt{41}$

$\vec{CB} = \begin{pmatrix}6\\-8\end{pmatrix}$

$|\vec{CB}| = \sqrt{(6)^2 + (-8)^2}$

$= 10$

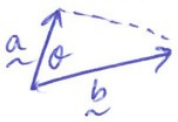
$$\cos\theta = \frac{(\sqrt{41})^2 + (10)^2 - (\sqrt{13})^2}{2(10)(\sqrt{41})}$$

$$= \frac{32}{5\sqrt{41}}$$

$$\theta = \cos^{-1}\left(\frac{32}{5\sqrt{41}}\right)$$

$$\approx 1°47'$$

Comment



when finding the angle between two vectors they must share the same tail

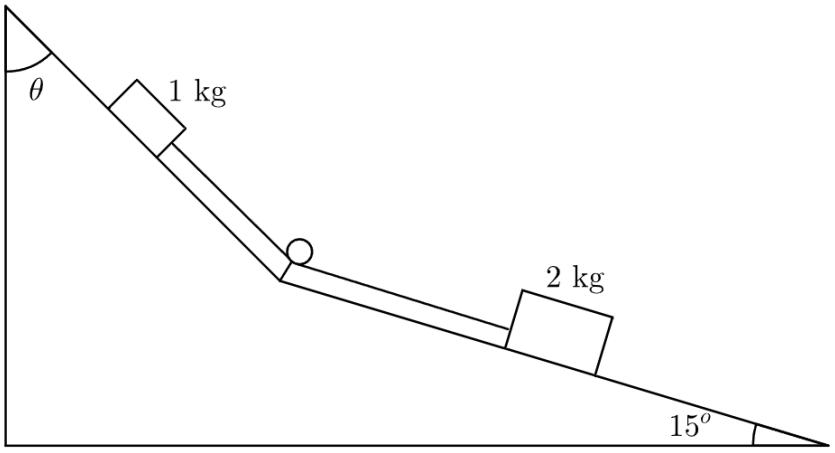


If $-\underset{\sim}{b}$ is used $180-\theta$ would be found instead.

Some students found $\angle CAB$ instead of $\angle BCA$ without knowing.
Care needs to be taken.

End of Q11 Solutions

(a) Two blocks, of mass 2 kg and 1 kg are initially stationary on a frictionless ramp as shown. A light, inextensible rope joins the two masses and is hooked under a frictionless pulley. The masses are then released.



(i) Assuming the rope is taut, find the acceleration of the 1 kg mass, leaving your answer in terms of g and θ. **2**

$$\begin{aligned} a &= g\cos\theta + T \\ 2a &= 2g\sin 15^\circ - T \\ a &= \frac{g}{3}\left(\cos\theta + 2\sin 15^\circ\right) \end{aligned}$$

Note: This was generally poorly done.

(ii) Find the range of values of θ such that the rope remains taut. **2**

From i) $$T = \frac{g}{3}\left(\cos\theta + 2\sin 15^\circ\right) - \frac{g}{3}\times 3\cos\theta$$
$$= \frac{2g}{3}\left(\sin 15^\circ - \cos\theta\right)$$

Rope remains taut when $\sin 15^\circ - \cos\theta > 0 \Rightarrow \theta > 75^\circ$

Note: Pupils stated $a_1 > a_2$, however their working shows $F_1 > F_2$ which does not give the correct response. This was also generally poorly done.

(b) (i) Find $\int (xe^x + e^x)\, dx$. **1**

$$
\begin{aligned}
&= \int \left(x\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x) \right) dx \\
&= \int \frac{d}{dx}\left(xe^x\right) dx \\
&= xe^x + C
\end{aligned}
$$

Note: Well done.

(ii) Hence, or otherwise, find $\int \frac{x+1}{\sqrt{e^{-2x} - x^2}} dx$. **2**

$$
\begin{aligned}
&= \int \frac{e^x (x+1)}{\sqrt{1 - e^{2x}x^2}} dx \\
&= \int \frac{e^x (x+1)}{\sqrt{1 - \left(xe^x\right)^2}} de \\
&= \sin^{-1} xe^x + C
\end{aligned}
$$

Note: Most pupils got this but quite a few didn't recognise this is an inverse trig solution.

(c) A particle moves horizontally in a line on the x-axis. It initially is moving at a speed of u m/s away from the origin such that $u > 1$. It experiences a pushing force numerically equal to $\frac{mk}{v}$ N due to the ground. There is also drag numerically equal to mkv N acting on the particle.

(i) Show that $\frac{d^2x}{dt^2} = k\left(\frac{1}{v} - v\right)$. **1**

$$ma = \frac{mk}{v} - mkv$$

$$\therefore \frac{d^2x}{dt^2} = k\left(\frac{1}{v} - v\right)$$

Note: Well done.

(ii) Prove that the time taken for the particle to stop is $T = \frac{1}{2k}\ln(u^2 - 1)$. **3**

$$\frac{dv}{dt} = k\left(\frac{1 - v^2}{v}\right)$$

$$-\frac{1}{2}\int_u^0 \frac{-2v}{1 - v^2}dv = k\int_0^T dt$$

$$\int_0^u \frac{-2v}{1 - v^2}dv = 2k\int_0^T dt$$

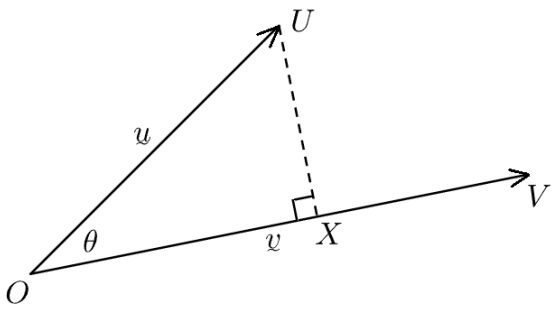
$$\left[\ln\left|1 - v^2\right|\right]_0^u = [2kt]_0^T$$

$$\ln\left|1 - u^2\right| - \ln\left(1 - 0^2\right) = 2k\left(T - 0\right)$$

$$T = \frac{1}{2k}\ln\left(u^2 - 1\right), u > 1$$

Note: Well done.

(d) Let U and V have position vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively.



(i) Show that $\overrightarrow{XU} = \underset{\sim}{u} - \text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$. **1**

$$\overrightarrow{OU} = \overrightarrow{OX} + \overrightarrow{XU}$$
$$\overrightarrow{XU} = \overrightarrow{OU} - \overrightarrow{OX}$$
$$\overrightarrow{XU} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}, \quad \text{where} \quad \overrightarrow{OX} = \text{proj}_{\mathbf{v}}\mathbf{u} \quad \text{and} \quad \overrightarrow{OU} = \mathbf{u}.$$

Note: Well done.

(ii) Hence, or otherwise, prove that $||\underset{\sim}{v}|\underset{\sim}{u} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\hat{\underset{\sim}{v}}| = |\underset{\sim}{u}||\underset{\sim}{v}| \sin\theta$ where θ is the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

$$\begin{aligned}
\sin\theta &= \frac{\left|\overrightarrow{XU}\right|}{\left|\overrightarrow{OU}\right|} \\
&= \frac{|\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}|}{|\mathbf{u}|} \\
|\mathbf{u}| \sin\theta &= |\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}| \\
&= \left|\mathbf{u} - \frac{(\mathbf{u} \cdot \mathbf{v})}{|\mathbf{v}|}\hat{\mathbf{v}}\right| \\
&= \frac{||\mathbf{v}|\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\,\hat{\mathbf{v}}|}{|\mathbf{v}|} \\
|\mathbf{v}||\mathbf{u}| \sin\theta &= ||\mathbf{v}|\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\,\hat{\mathbf{v}}|
\end{aligned}$$

Note: Mostly well done.

End of Q12 Solutions

(a) For all positive x, a function f satisfies

$$2f(x) - f\left(\frac{1}{x}\right) = \frac{1}{x} + x\ln x. \quad (1)$$

(i) Explain why $2f\left(\frac{1}{x}\right) - f(x) = x - \frac{\ln x}{x}$ (2) **1**

Replace $x \leftrightarrow \frac{1}{x}$

$$2f\left(\frac{1}{x}\right) - f\left(\frac{1}{\frac{1}{x}}\right) = \frac{1}{\frac{1}{x}} + \frac{1}{x}\ln\frac{1}{x}$$

$$2f\left(\frac{1}{x}\right) - f(x) = x + \frac{1}{x}\ln x^{-1}$$

$$\therefore\ 2f\left(\frac{1}{x}\right) - f(x) = x - \frac{\ln x}{x}$$

Generally well done.

(ii) Evaluate $\int_1^2 f(x)\,dx$. **4**

(1) × 2 : $4f(x) - 2f\left(\frac{1}{x}\right) = \frac{2}{x} + 2x\ln x$ (3)

(2) + (3) : $3f(x) = \frac{2}{x} + 2x\ln x + x - \frac{\ln x}{x}$

$$f(x) = \frac{1}{3}\left[\frac{2}{x} + 2x\ln x + x - \frac{\ln x}{x}\right]$$

Consider $2\int x\ln x\,dx = 2\left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right]$

$= x^2\ln x - \frac{x^2}{2} + C$

$u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

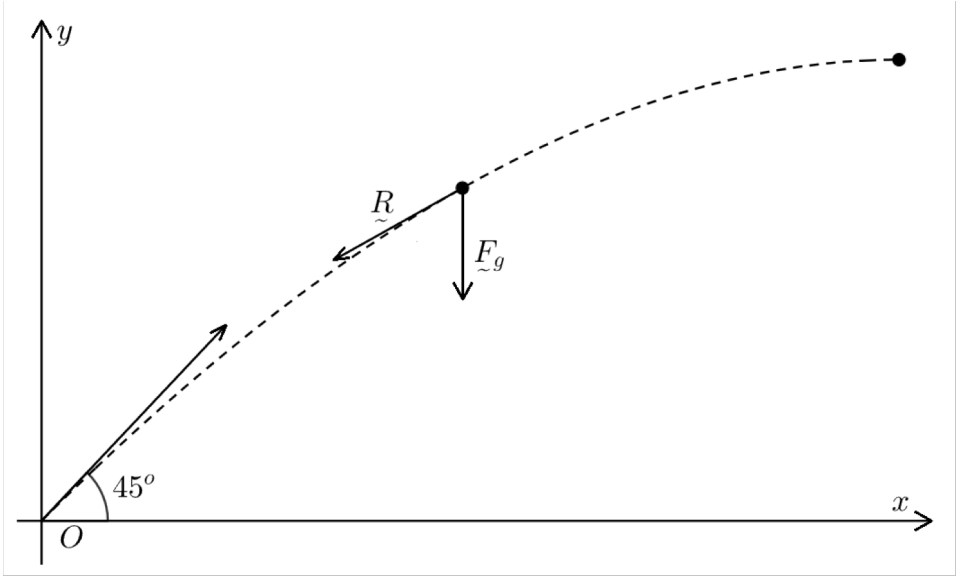
$$\int_1^2 f(x)\,dx = \frac{1}{3}\left[2\ln|x| + x^2\ln x - \frac{x^2}{2} + \frac{x^2}{2} - \frac{(\ln x)^2}{2}\right]_1^2$$

$$= \frac{1}{3}\left[2\ln 2 + 4\ln 2 - \frac{(\ln 2)^2}{2}\right]$$

$$= 2\ln 2 - \frac{1}{6}(\ln 2)^2$$

- Many algebraic mistakes, mainly to do with negatives.
- It was difficult to award marks if students couldn't even find $f(x)$.

(b) A 5 g flare is shot at 45^o with speed $10\sqrt{2}$ from the origin and is programmed to detonate when it reaches its maximum height.



There is air resistance numerically proportional to the velocity squared, $\underset{\sim}{R} = -k\begin{pmatrix}\dot{x}^2\\ \dot{y}^2\end{pmatrix}$, acting on the flare, where $\dot{x}$ and $\dot{y}$ are the horizontal and vertical velocities of the flare respectively. Let $k = 2.5 \times 10^{-4}$ and $g = 9.8$ m/s^2.

(i) Show that the acceleration vector $\underset{\sim}{a}$ of the particle is $\underset{\sim}{a} = \begin{pmatrix}-0.05\dot{x}^2\\ -9.8 - 0.05\dot{y}^2\end{pmatrix}$. **1**

$m = 5g = 0.005\,kg$

$\uparrow +$

$\downarrow$ mg $\quad \downarrow$ $k\dot{y}^2$

$m\underset{\sim}{a} = \begin{pmatrix}-k\dot{x}^2\\ -9.8m - k\dot{y}^2\end{pmatrix}$

$\underset{\sim}{a} = \begin{pmatrix}-\frac{k}{m}\dot{x}^2\\ -9.8 - \frac{k}{m}\dot{y}^2\end{pmatrix}$ $\qquad \frac{k}{m} = \frac{2.5\times10^{-4}}{0.005} = 0.05$

$\therefore \underset{\sim}{a} = \begin{pmatrix}-0.05\dot{x}^2\\ -9.8 - 0.05\dot{y}^2\end{pmatrix}$

- Generally well done.
- Students need to explicitly have $m = 0.005$.

(b) (continued)

(ii) Prove that $x = 20\ln\left(1+\frac{t}{2}\right)$. **3**

Method 1

$\ddot{x} = -0.05\dot{x}^2 = \frac{d\dot{x}}{dt}$

$\int_{10\sqrt{2}\cos 45°}^{\dot{x}} \frac{1}{\dot{x}^2}\, d\dot{x} = -\int_0^t 0.05\, dt$

$\left[-\frac{1}{\dot{x}}\right]_{10}^{\dot{x}} = -\left[0.05t\right]_0^t$

$\frac{1}{\dot{x}} - \frac{1}{10} = \frac{1}{20}t$

$\frac{1}{\dot{x}} = \frac{1}{10} + \frac{1}{20}t = \frac{t+2}{20}$

$\dot{x} = \frac{20}{t+2} = \frac{dx}{dt}$

$\int_0^x dx = \int_0^t \frac{20}{t+2}\, dt$

$x = 20\left[\ln|t+2|\right]_0^t$

$= 20\left[\ln(t+2) - \ln 2\right] \quad (t+2>0)$

$= 20\ln\left(\frac{t+2}{2}\right)$

$= 20\ln\left(1+\frac{t}{2}\right)$

Method 2

$\ddot{x} = -0.05\dot{x}^2 = \frac{\dot{x}\,d\dot{x}}{dx}$

$\int_{10\sqrt{2}\cos 45°}^{\dot{x}} \frac{d\dot{x}}{\dot{x}^2} = \int_0^x -0.05\, dx$

$\left[\ln|\dot{x}|\right]_{10}^{\dot{x}} = \left[-0.05x\right]_0^x$

$\ln|\dot{x}| - \ln 10 = -0.05x$

$\ln\left|\frac{\dot{x}}{10}\right| = -0.05x$

$\dot{x} = 10e^{-0.05x} = \frac{dx}{dt} \quad (x=0,\ \dot{x}=10)$

$\int_0^x e^{0.05x}\, dx = \int_0^t 10\, dt$

$20\left[e^{0.05x}\right]_0^x = \left[10t\right]_0^t$

$e^{0.05x} - 1 = \frac{t}{2}$

$e^{0.05x} = 1 + \frac{t}{2}$

$0.05x = \ln\left(\frac{t}{2}+1\right)$

$x = 20\ln\left(\frac{t}{2}+1\right)$

- Generally well done.
- Method 1 was more popular.
- Some students had initial $\dot{x} = 10\sqrt{2}\sin 45°$, which luckily didn't impact the proof.

(b) (continued)

(iii) Prove that during the upwards motion, $\dot{y} = \dfrac{14(5 - 7\tan(0.7t))}{7 + 5\tan(0.7t)}$. **2**

$\ddot{y} = -9.8 - 0.05\dot{y}^2 = \dfrac{d\dot{y}}{dt}$

$\displaystyle\int_{10\sqrt{2}\sin 45°}^{\dot{y}} \frac{d\dot{y}}{9.8 + 0.05\dot{y}^2} = \int_0^t -dt$

$\displaystyle\int_{10}^{\dot{y}} \frac{20\,d\dot{y}}{196 + \dot{y}^2} = [-t]_0^t$

$\dfrac{20}{14}\left[\tan^{-1}\dfrac{\dot{y}}{14}\right]_{10}^{\dot{y}} = -t$

$\dfrac{10}{7}\left[\tan^{-1}\dfrac{\dot{y}}{14} - \tan^{-1}\dfrac{10}{14}\right] = -t$

$\tan^{-1}\dfrac{\dot{y}}{14} = -\dfrac{7}{10}t + \tan^{-1}\dfrac{5}{7}$

$\dfrac{\dot{y}}{14} = \tan\left(\tan^{-1}\dfrac{5}{7} - \dfrac{7}{10}t\right)$

$= \dfrac{\frac{5}{7} - \tan(0.7t)}{1 + \frac{5}{7}\tan(0.7t)}$

$= \dfrac{5 - 7\tan(0.7t)}{7 + 5\tan(0.7t)}$

$\dot{y} = 14\left[\dfrac{5 - 7\tan(0.7t)}{7 + 5\tan(0.7t)}\right]$

- Poorly done.
- Many students could integrate, but would not complete the proof.
- ALL working needs to be shown for PROVE questions.

(b) (continued)

(iv) Hence, find the time until detonation, to 2 decimal places. **1**

Detonation when maximum height $\dot{y} = 0$

$$14\left[\frac{5-7\tan(0.7t)}{7+5\tan(0.7t)}\right] = 0$$

$$5-7\tan(0.7t) = 0$$

$$\tan(0.7t) = \frac{5}{7}$$

$$0.7t = \tan^{-1}\frac{5}{7}$$

$$t = \frac{1}{0.7}\tan^{-1}\frac{5}{7}$$

$$= 0.89 \text{ s}$$

- This question was supposed to be an easy mark, especially for those who could not do previous parts.
- MANY (a third of the cohort) students had their calculator in degrees mode.

(b) (continued)

(v) Prove that the vertical displacement of the flare is **3**

$$y = 20 \ln \left| \cos(0.7t) + \frac{5}{7} \sin(0.7t) \right| .$$

$$\dot{y} = 14 \left[\frac{5 - 7\tan(0.7t)}{7 + 5\tan(0.7t)} \right] = \frac{dy}{dt}$$

$$\int_0^y dy = 14 \int_0^t \frac{5 - 7\tan(0.7t)}{7 + 5\tan(0.7t)} \, dt \quad \times \quad \frac{\cos(0.7t)}{\cos(0.7t)}$$

$$y = 14 \int_0^t \frac{5\cos(0.7t) - 7\sin(0.7t)}{7\cos(0.7t) + 5\sin(0.7t)} \, dt$$

$$= \frac{14}{0.7} \int_0^t \frac{0.7\left[-7\sin(0.7t) + 5\cos(0.7t)\right]}{7\cos(0.7t) + 5\sin(0.7t)} \, dt$$

$$= 20 \int_0^t \frac{0.7\left[-\sin(0.7t) + \frac{5}{7}\cos(0.7t)\right]}{\cos(0.7t) + \frac{5}{7}\sin(0.7t)} \, dt$$

$$= 20 \left[\ln \left| \cos(0.7t) + \frac{5}{7}\sin(0.7t) \right| \right]_0^t$$

$$= 20 \left[\ln \left| \cos(0.7t) + \frac{5}{7}\sin(0.7t) \right| - \ln 1 \right]$$

$$= 20 \ln \left| \cos(0.7t) + \frac{5}{7}\sin(0.7t) \right|$$

- Again, PROVE questions = show ALL working.
- Many wrote: $y = 14 \int_0^t \frac{5\cos(0.7t) - 7\sin(0.7t)}{7\cos(0.7t) + 5\sin(0.7t)} \, dt$

  $= 20 \left[\ln \left| \cos(0.7t) + \frac{5}{7}\sin(0.7t) \right| \right]_0^t$

  This is not showing ALL working.

(b) (continued)

(vi) Hence, find the position at which the flare will detonate, to 2 decimal places. **1**

$t = \frac{1}{0.7}\tan^{-1}\frac{5}{7}$ or 0.89

$x = 7.33$ m or 7.36 m (using $t = 0.89$)

$y = 4.12$ m

- Another easy mark to get, assuming students found a value of t prior.
- Some students only found x or y, and not both.

End of Q13 Solutions

(a) Let $I_n = \int_a^1 x^2(\ln x)^n \, dx$, where $a > 0$ and n is a non-negative integer.

(i) Prove that $I_n = -\frac{a^3(\ln a)^n}{3} - \frac{n}{3} I_{n-1}$, and state the values of n for which this is true. **4**

Let $u = (\ln x)^n$ $\quad \frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{n}{x}(\ln x)^{n-1}$ $\quad v = \frac{x^3}{3}$

$\therefore I_n = \left[\frac{x^3}{3}(\ln x)^n\right]_a^1 - \int_a^1 \frac{x^3}{3}\left(\frac{n}{x}(\ln x)^{n-1}\right) dx$

$= \left[\left(\frac{1^3}{3}(\ln 1)^n - \frac{a^3}{3}(\ln a)^n\right)\right] - \frac{n}{3}\int_a^1 x^2(\ln x)^{n-1} dx$

As $\ln 1 = 0$ and $I_{n-1} = \int_a^1 x^2(\ln x)^{n-1} dx$

$\therefore = 0 - \frac{a^3}{3}(\ln a)^n - \frac{n}{3} I_{n-1}$

$= \frac{a^3}{3} - \frac{n}{3} I_{n-1}$

$\because I_n$ is defined for $n \geq 0$

For I_{n-1} to exist, $n-1 \geq 0$

$\therefore$ Formula is true for $n \geq 1$

Marking scheme:

1 mark: Correct setup of the pairs of Integration by Parts components (u, u' and v' and v) or equivalent.

1 mark: Correctly obtains $I_n = \left[\frac{x^3}{3}(\ln x)^n\right]_a^1 - \frac{n}{3}\int_a^1 x^2(\ln x)^{n-1} dx$.

1 mark: Correctly substitutes boundary limits, explicitly showing the upper bound vanishes because $\ln(1) = 0$.

1 mark: Correctly states the restriction $n \geq 1$.

Marker's comments:

- Majority of candidates performed well in this question.
- Some candidates were penalised for not stating the required restriction on the parameter at the end of their solution, or for omitting necessary working in a higher-mark question.

(a) (continued)

(ii) Define $J_n = \lim_{a \to 0} I_n$. **4**

Hence, deduce that $J_n = \dfrac{(-1)^n n!}{3^{n+1}}$ for $n \geq 0$.

$J_n = \lim_{a \to 0} I_n$

$= \lim_{a \to 0} \dfrac{-a^3 (\ln a)^n}{3} - \dfrac{n}{3} \lim_{a \to 0} I_{n-1}$

Now $\lim_{a \to 0} a^3 (\ln a)^n = 0^+$ (polynomial function (a^3) approach 0 at much faster rate than log function $(\ln a)^n$ approach either ∞ or $-\infty$ when $a \to 0$)

$\therefore J_n = 0 - \dfrac{n}{3} J_{n-1}$

$= -\dfrac{n}{3} J_{n-1}$

$= \left(-\dfrac{n}{3}\right)\left(-\dfrac{n-1}{3}\right) J_{n-2}$ (Using recurrence relation

$= \left(\dfrac{-n}{3}\right)\left(-\dfrac{n-1}{3}\right)\left(-\dfrac{n-2}{3}\right) J_{n-3}$

$\vdots$

$= \underbrace{\left(-\dfrac{n}{3}\right)\left(-\dfrac{n-1}{3}\right)\left(-\dfrac{n-2}{3}\right) \cdots \left(-\dfrac{1}{3}\right)}_{n \text{ terms}} J_0$

$= \dfrac{(-1)^n [n(n-1)(n-2) \cdots \times 1]}{3^n} \times J_0$

Now $J_0 = \lim_{a \to 0} \int_a^1 x^2 \, dx$

$= \lim_{a \to 0} \left[\dfrac{x^3}{3}\right]_a^1$

$= \lim_{a \to 0} \left(\dfrac{1}{3} - \dfrac{a}{3}\right)$

$= \dfrac{1}{3} - \dfrac{0}{3}$

$= \dfrac{1}{3}$

$\therefore J_n = \dfrac{(-1)^n \times n!}{3^n} \times \dfrac{1}{3} = \dfrac{(-1)^n n!}{3^{n+1}}$

(a) (ii) (continued)

Marking scheme:

1 mark: Correctly obtains the recurrence for J_n from part (a) (i), $J_n = -\frac{n}{3}J_{n-1}$, with a brief justification that $\lim_{a\to 0} a^3(\ln a)^n = 0$.

1 mark: Iterates the recurrence relation correctly to find the expression in terms of J_0.

1 mark: Correctly finds $J_0 = \frac{1}{3}$ using integration.

1 mark: Show why the required result has the $(-1)^n$, $n!$ and 3^{n+1} using the pattern from the recurrence relation.

Marker's comments:

This part was not well answered by many candidates.
Common issues included:

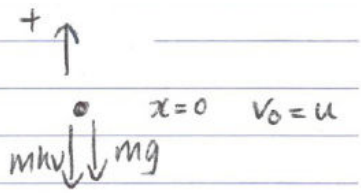
- Many candidates did not justify why $\lim_{a\to 0} a^3(\ln a)^n = 0$.
  A significant number of candidates wrote that because $a \to 0$, the entire expression must tend to 0, without considering that $\ln a \to -\infty$ as $a \to 0^+$.
  Hence, the expression has the indeterminate form $0 \times \pm\infty$, so some brief justification was required.

- Some candidates incorrectly treated the recurrence as a geometric progression, despite the multiplier changing at each iteration.

- Many candidates did not iterate the recurrence sufficiently to demonstrate the general pattern leading to the $n!, (-1)^n$ *or* 3^{n+1} factors

- As this was a **show** question, candidates needed to demonstrate the required result from the recurrence and limits rather than assume, work backwards from, or simply state the given expression.

(b) A mass m kg is shot upwards at initial speed u m/s in a medium that applies a resistive force numerically equal to mkv N, where $k > 0$ is the constant of proportionality and v is the velocity of the mass in m/s. The acceleration due to gravity is g m/s^2. Take upwards as positive. Prove that the object's displacement satisfies $v - u + gt + kx = 0$. **4**



$ma = -mg - mkv$

$a = -g - kv$ (*)

$\frac{dv}{dt} = -g - kv$

$\frac{dt}{dv} = -\frac{1}{g+kv}$

$\int_0^t dt = \int_u^v -\frac{1}{g+kv}\, dv$

$= -\frac{1}{k}\int_u^v \frac{k}{g+kv}\, dv$

$t - 0 = -\frac{1}{k}\left[\ln|g+kv|\right]_u^v$

$t = -\frac{1}{k}\left(|\ln g+kv| - |\ln g+ku|\right)$

$-kt = \ln\left|\frac{g+kv}{g+ku}\right|$ (1)

Now $v\frac{dv}{dx} = -g - kv$ (Method 1)

$\frac{dv}{dx} = -\frac{(g+kv)}{v}$

$\frac{dx}{dv} = -\frac{v}{g+kv}$

$= -\frac{1}{k}\left(\frac{(g+kv)-g}{g+kv}\right)$

$= -\frac{1}{k}\left(1 - \frac{g}{g+kv}\right)$

$\int_0^x dx = -\frac{1}{k}\int_u^v 1 - \frac{g}{k}\times\frac{k}{g+kv}\, dv$

$x - 0 = -\frac{1}{k}\left[v - \frac{g}{k}\ln|g+kv|\right]_u^v$

$-kx = v - \frac{g}{k}\ln|g+kv| - u + \frac{g}{k}\ln|g+ku|$

(b) (continued)

$$\therefore\ v - u - \frac{g}{k}\ln\left|\frac{g+kv}{g+ku}\right| + kx = 0 \quad (2)$$

Sub (1) into (2)

$$v - u - \frac{g}{k}(-kt) + kx = 0$$

$$\therefore\ v - u + gt + kx = 0$$

OR (Method 2)

From (1)

$$-kt = \ln\left|\frac{g+kv}{g+ku}\right|$$

$$-kt = \ln\left(\frac{g+kv}{g+ku}\right)$$

As $a < 0$ (From $*$)

$g + kv > 0$

$\therefore\ g + ku > 0$

$\therefore\ \frac{g+kv}{g+ku} > 0$

$$\frac{g+kv}{g+ku} = e^{-kt} \quad (\#)$$

$$v = \left(\frac{g+ku}{k}\right)e^{-kt} - \frac{g}{k}$$

$$\therefore\ \frac{dx}{dt} = \left(\frac{g+ku}{k}\right)e^{-kt} - \frac{g}{k}$$

$$\int_0^x dx = \int_0^t \left(\frac{g+ku}{k}\right)e^{-kt} - \frac{g}{k}\,dt$$

$$x - 0 = \left[-\left(\frac{g+ku}{k^2}\right)e^{-kt} - \frac{gt}{k}\right]_0^t$$

$$= -\left(\frac{g+ku}{k^2}\right)e^{-kt} - \frac{gt}{k} + \left(\frac{g+ku}{k^2}\right)e^{-k0} + \frac{g(0)}{k}$$

$$x = -\left(\frac{g+ku}{k^2}\right)e^{-kt} - \frac{gt}{k} + \left(\frac{g+ku}{k^2}\right)$$

$$\therefore\ kx = -\frac{(g+ku)}{k}e^{-kt} - gt + \frac{g+ku}{k} \quad (3)$$

$$kx + gt = -\left(\frac{g+ku}{k}\right)x\left(\frac{g+kv}{g+ku}\right) + \frac{g}{k} + u \quad (\text{Sub } \# \text{ into } 3)$$

$$kx + gt = -\frac{g}{k} - v + \frac{g}{k} + u$$

$$\therefore\ v - u + gt + kx = 0$$

(b) (continued)

Marking scheme:

1 mark: Identifying $a = -g - kv$

1 mark: Correctly uses $\frac{dv}{dt} = -g - kv$ and the initial conditions to obtain a correct expression for in terms of u, v, k and g.

1 mark: Correctly uses $v\frac{dv}{dx} = -g - kv$ or the expression for t and the initial conditions to obtain a correct expression for x.

1 mark: Correct substitution of expressions to obtain the required result.

Marker's comments:

- This question was generally well answered by candidates.
- Some candidates made algebraic or integration errors and then fudged their working to arrive at the required result, rather than obtaining it logically from their preceding work.
- A common error was omitting the absolute value when integrating $\frac{1}{g + kv}$ without justification. Candidates needed to briefly justify why $g + kv > 0$ over the relevant motion or simply use absolute value.

(c) The probability density function $\phi(z)$ of a standard normal variable $Z \sim N(0,1)$ is given by $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, for $-\infty < z < \infty$.

You may assume that $E(Z^n) = \int_{-\infty}^{\infty} z^n \phi(z)\, dz = \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} z^n \phi(z)\, dz$.

(i) Explain why $E(Z^2) = 1$. **1**

$\text{Var}(z) = E(z^2) - [E(z)]^2$

$\therefore E(z^2) = \text{Var}(z) + [E(z)]^2$

$= 1 + 0^2$ (as $z \sim N(0,1)$)

$= 1$

Marking scheme:

1 mark: Correct solution using the result given in the reference Sheet.

Marker's comments:

- This question was poorly answered by the majority of the candidates.
- Many candidates stated that the result was because the normal distribution has a PDF/CDF whose total probability is 1. This does not justify $E(Z^2) = 1$, since the expectation involves the additional factor z^2.
- Some candidates attempted to evaluate the expectation directly by integration but most of those were unable to complete the integral successfully.
  Candidates attempting lengthy integration should have taken note that this was a 1-mark question, indicating that a short application of a known result was expected.
- **Although this result is from Mathematics Advanced, all Advanced content is assumed knowledge and may be assessed (not as a stand-alone question) within the Extension 2 examination.**
  This was stated in the exam notification.

(c) (continued)

(ii) Hence, or otherwise, prove that $E(Z^4) = 3$. **3**

$$E(z^4) = \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} z^4 \phi(z)\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} z^4 e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} z^3 \frac{d\left(-e^{-\frac{z^2}{2}}\right)}{dz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \left[-z^3 e^{-\frac{z^2}{2}}\right]_{-\varepsilon}^{\varepsilon} - \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \int_{-\varepsilon}^{\varepsilon} -3z^2 e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \left[\left(-\varepsilon^3 e^{-\frac{\varepsilon^2}{2}} - \left(-(-\varepsilon)^3 e^{-\frac{(-\varepsilon)^2}{2}}\right)\right)\right] + 3\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} z^2\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to \infty} \left(-2\varepsilon e^{-\frac{\varepsilon^2}{2}}\right) + 3E(z^2)$$

$$= \frac{1}{\sqrt{2\pi}} \times 0 + 3E(z^2)$$

(As exponential terms goes to 0 faster than linear term goes to ∞ as $\varepsilon \to \infty$)

$$= 3 \cdot (1) \quad \text{(From (i))}$$

$$= 3$$

Marking scheme:

1 mark: Correct setup of the pairs of Integration by Parts components (u, u' and v' and v) or equivalent.

1 mark: Correctly obtains $E(Z^4) = \frac{1}{\sqrt{2\pi}}\left(\lim_{\varepsilon\to\infty}\left[-z^3 e^{-\frac{z^2}{2}}\right]_{-\varepsilon}^{\varepsilon} - \lim_{\varepsilon\to\infty}\int_{-\varepsilon}^{\varepsilon} -3z^2 e^{-\frac{z^2}{2}} dz\right)$ and thereby identifying the second term as 3E (Z^2).

1 mark: Correctly evaluates the first term as 0 using limits, with appropriate justification and, use the result from (i) to obtain to evaluate 3E (Z^2) to obtain the required result.

(c) (ii) (continued)

Marker's comments:

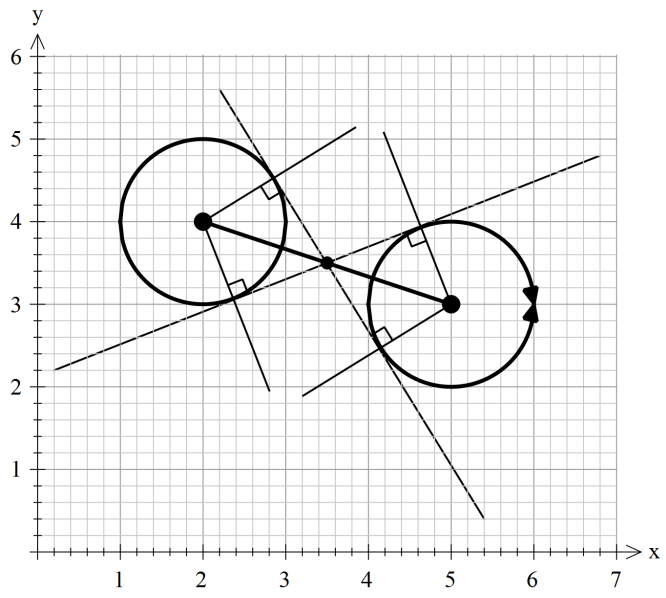
- This part was poorly answered by most candidates.
- Many candidates chose an inappropriate pairing for integration by parts or made errors when carrying out the integration.
- Some candidates made no attempt to use Integration by Parts, or any appropriate Extension 2 technique. Instead, some relied on false or unsupported statements to try to obtain the required result.
- Candidates needed to retain and evaluate the limits provided in the question.
  Infinity cannot be treated as ordinary endpoint substitutions, since infinity is not a real number.
  The improper integral must be interpreted through a limit as the finite bounds tend to infinity which was given in the question.
- When showing that the resulting term tends to 0, candidates were expected to provide a brief justification rather than simply assert the result.

End of Q14 Solutions

(a) Let z, w be complex numbers that satisfy **4**

$$|z-5-3i|=1 \qquad \text{and} \qquad |w-2-4i|=1.$$

Find the range of values of $\text{Arg}\,(z-w)$, where Arg denotes the principal argument.



The angle of the line segment between centres is

$$\tan^{-1}\left(\frac{4-3}{2-5}\right) = -\tan^{-1}\frac{1}{3}$$

The midpoint is $\left(\frac{2+5}{2}, \frac{4+3}{2}\right) = \left(\frac{7}{2}, \frac{7}{2}\right)$

The distance from centre to mid point is

$$2d = \sqrt{(4-3)^2+(2-5)^2}$$

$$d = \frac{\sqrt{10}}{2}$$

The angle of the tangents to the line segment is $\sin^{-1}\frac{\sqrt{10}}{5}$, since the radii are 1.

We want the range of values that are a rotation around the midpoint by the angle stated above.

$$\therefore -\sin^{-1}\frac{\sqrt{10}}{5} - \tan^{-1}\frac{1}{3} < Arg\,(z-w) < \sin^{-1}\frac{\sqrt{10}}{5} - \tan^{-1}\frac{1}{3}$$

Note: Very poorly done. Only a hand full of pupils got full marks.

(b) Let n points $P_1, P_2, \ldots, P_n$, which are represented by the position vectors $\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots, \underset{\sim}{p}_n$ respectively, lie on a straight line l. **4**

Prove by mathematical induction, for $n \geq 2$, that the point represented by

$$\underset{\sim}{u} = \lambda_1 \underset{\sim}{p}_1 + \lambda_2 \underset{\sim}{p}_2 + \cdots + \lambda_n \underset{\sim}{p}_n$$

is collinear with the n points, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.

Prove true for the base case.

$$\begin{aligned}\mathbf{u}_2 &= \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2, \alpha_1 + \alpha_2 = 1. \\ &= (1 - \alpha_2)\, \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 \\ &= \mathbf{p}_1 + \alpha_2 (\mathbf{p}_2 - \mathbf{p}_1)\end{aligned}$$

Therefore the point represented by $\mathbf{u}_2$ is collinear with the points represented by $\mathbf{p}_1$ and $\mathbf{p}_2$.

Assume true for $n = k$. $\mathbf{u}_k = \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 + \ldots + \lambda_k \mathbf{p}_k, \sum_{i=1}^{k} \lambda_i = 1$

and the point represented by $\mathbf{u}_k$ is collinear with all points represented by $\mathbf{p}_i$.

Prove true for $n = k + 1$.

$$\mathbf{u}_{k+1} = \mu_1 \mathbf{p}_1 + \mu_2 \mathbf{p}_2 + \ldots + \mu_k \mathbf{p}_k + \mu_{k+1} \mathbf{p}_{k+1}, \sum_{i=1}^{k+1} \mu_i = 1$$

$$\mathbf{u}_{k+1} = (1 - \mu_{k+1}) \left[\frac{\mu_1}{1 - \mu_{k+1}} \mathbf{p}_1 + \frac{\mu_2}{1 - \mu_{k+1}} \mathbf{p}_2 + \ldots + \frac{\mu_k}{1 - \mu_{k+1}} \mathbf{p}_k \right] + \mu_{k+1} \mathbf{p}_{k+1}$$

Let $\lambda_i = \dfrac{\mu_i}{1 - \mu_{k+1}}$, $\displaystyle\sum_{i=1}^{k} \frac{\mu_i}{1 - \mu_{k+1}} = \frac{1}{1 - \mu_{k+1}} \sum_{i=1}^{k} \mu_i = \frac{1}{1 - \mu_{k+1}} (1 - \mu_{k+1}) = 1 = \sum_{i=1}^{k} \lambda_i$

$$\begin{aligned}\mathbf{u}_{k+1} &= (1 - \mu_{k+1})\, \mathbf{u}_k + \mu_{k+1} \mathbf{p}_{k+1} \\ &= \mathbf{u}_k + \mu_{k+1} (\mathbf{p}_{k+1} - \mathbf{u}_k)\end{aligned}$$

Therefore the point represented by $\mathbf{u}_{k+1}$ is collinear with the points represented by $\mathbf{u}_k$ and $\mathbf{p}_{k+1}$.

The statement is true for the base case, and true for $n = k + 1$, given $n = k$ is true. Therefore it is true for all n greater or equal to 2.

(b) (continued)

Note: No-one got full marks for this. In fact, only a handful of pupils successfully proved the base case.

(c) (i) Let a, b and c be real numbers. Prove that if $a\sin x + b\cos x = c$ has no solutions for all real x, then $a^2 + b^2 < c^2$. **2**

$$R\sin(x+\alpha) = R(\sin x\cos\alpha + \cos x\sin\alpha)$$

Comparing coefficients: $R\cos\alpha = a, R\sin\alpha = b$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = a^2 + b^2$$
$$\Rightarrow R^2 = a^2 + b^2$$
$$\Rightarrow R = \sqrt{a^2+b^2}$$

Therefore R is the amplitude of $a\sin x + b\cos x$.

$\sin(x+\alpha) = \dfrac{c}{R}$ has no real solutions when

$$\left|\frac{c}{R}\right| > 1, \text{since} -1 < \sin\theta < 1.$$

$$\Rightarrow |c| > |R|$$
$$\Rightarrow c^2 > a^2 + b^2$$

Note: Mostly well done.

(c) (continued)

(ii) Particle P is released from rest 4 m to the right of the origin, and follows motion governed by the equation $\ddot{x}_1 = 8 - 4x_1$. **4**

Particle Q undergoes simple harmonic motion centred at the origin. Q is released T seconds later from the origin, with the same period and amplitude as particle P and negative velocity.

For what range of values of T during particle P's first oscillation should Q be released so that the two particles never meet?

$$\ddot{x}_1 = -4(x_1 - 2) \quad \Rightarrow n = 2, c = 2.$$

$$x_1 = a\cos(2t + \alpha_1) + 2$$

P is at rest at its full amplitude and is at $x = 4$ $\Rightarrow a = 2, \alpha_1 = 0.$

$$x_1 = 2\cos 2t + 2$$

Q has same period and amplitude but starts at the origin with negative velocity and is centred at the origin.

$$\therefore x_2 = 2\cos\left(2t + \frac{\pi}{2}\right)$$

$$2\cos(2(t+T)) + 2 = 2\cos\left(2t + \frac{\pi}{2}\right)$$

$$\cos\left(2t + \frac{\pi}{2}\right) - \cos(2(t+T)) = 1$$

$$\cos 2t\cos\frac{\pi}{2} - \sin 2t\sin\frac{\pi}{2} - \cos 2t\cos 2T + \sin 2T\sin 2t = 1$$

$$\sin 2T\sin 2t - \sin 2t - \cos 2T\cos 2t = 1$$

$$(\sin 2T - 1)\sin 2t - \cos 2T\cos 2t = 1$$

From i) $$(\sin 2T - 1)^2 + (-\cos 2T)^2 < 1^2$$

$$\sin^2 2T - 2\sin 2T + 1 + \cos^2 2T < 1$$

$$2 - 2\sin 2T < 1$$

$$\sin 2T > \frac{1}{2}$$

$$2T > \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}^+ \cup \{0\}$$

$$2T \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) + 2k\pi, k \in \mathbb{Z}^+ \cup \{0\}$$

$$T \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$$

(c) (continued)

Note: Only 2 pupils got full marks for this question. Most pupils had some idea of the displacement equations, but were unable to successfully take it much further than that.

End of Q15 Solutions

(a) Let $\beta = \dfrac{-3+\sqrt{5}}{2}$.

(i) Show that $\beta^2 + 3\beta + 1 = 0$, and hence, prove that $\beta^{-2} + 3\beta^{-1} + 1 = 0$. **2**

Consider $x^2 + 3x + 1 = 0$

It has roots $x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$

$x = \dfrac{-3 \pm \sqrt{5}}{2}$

$\beta = \dfrac{-3+\sqrt{5}}{2}$ satisfies $x^2 + 3x + 1 = 0$

ie $\beta^2 + 3\beta + 1 = 0$.

$\alpha\beta = \dfrac{c}{a}$ (product of roots)

$\alpha\beta = \dfrac{1}{1}$

$\alpha = \dfrac{1}{\beta}$

the other root is $\dfrac{1}{\beta} = \dfrac{-3-\sqrt{5}}{2}$

$\dfrac{1}{\beta}$ also satisfies $x^2 + 3x + 1 = 0$

$\left(\dfrac{1}{\beta}\right)^2 + 3\left(\dfrac{1}{\beta}\right) + 1 = 0$

ie. $\beta^{-2} + 3\beta^{-1} + 1 = 0$

Alternatively,

$\beta = \dfrac{-3+\sqrt{5}}{2}$

$\beta^2 = \left(\dfrac{-3+\sqrt{5}}{2}\right)^2$

$= \dfrac{9 - 6\sqrt{5} + 5}{4}$

$= \dfrac{14 - 6\sqrt{5}}{4}$

$= \dfrac{7 - 3\sqrt{5}}{2}$

(a) (i) (continued)

$$\beta^2+3\beta+1 = \frac{7-3\sqrt{5}}{2} + 3\left(\frac{-3+\sqrt{5}}{2}\right) + 1$$

$$= \frac{7-3\sqrt{5}-9+3\sqrt{5}+2}{2}$$

$$= 0$$

$$\beta^2 + 3\beta + 1 = 0$$

$$\frac{\beta^2+3\beta+1}{\beta^2} = \frac{0}{\beta^2} \qquad \text{since } \beta \neq 0$$

$$1 + 3\beta^{-1} + \beta^{-2} = 0$$

i.e. $\beta^{-2}+3\beta^{-1}+1=0$

Comment:

Time and effort can be saved by not rushing into needless calculations and algebra.

(a) (continued)

(ii) Let $B_n = \beta^n + \beta^{-n}$. Deduce that $B_n = -3B_{n-1} - B_{n-2}$. **1**

$x^2 + 3x + 1 = 0$ has roots β, β^{-1}

$x^2 = -3x - 1$ has roots β, β^{-1}

$x^{n-2} \cdot x^2 = x^{n-2}(-3x - 1)$ has roots $0, \beta, \beta^{-1}$

$x^n = -3x^{n-1} - x^{n-2}$

$\beta^n = -3\beta^{n-1} - \beta^{n-2}$ ——— (1)

$(\beta^{-1})^n = -3(\beta^{-1})^{n-1} - (\beta^{-1})^{n-2}$

$\beta^{-n} = -3\beta^{-n+1} - \beta^{-n+2}$ ——— (2)

(1) + (2)

$\beta^n + \beta^{-n} = -3(\beta^{n-1} + \beta^{-n+1}) - (\beta^{n-2} + \beta^{-n+2})$

$\therefore B_n = -3B_{n-1} - B_{n-2}$

Alternatively,

consider $B_n + 3B_{n-1} + B_{n-2}$

$= \beta^n + \beta^{-n} + 3(\beta^{n-1} + \beta^{-n+1}) + \beta^{n-2} + \beta^{-n+2}$

$= \beta^n + 3\beta^{n-1} + \beta^{n-2} + \beta^{-n} + 3\beta^{-n+1} + \beta^{-n+2}$

$= \beta^{n-2}(\beta^2 + 3\beta + 1) + \beta^{-n}(1 + 3\beta + \beta^2)$

$= \beta^{n-2}(0) + \beta^{-n}(0)$

$= 0$

since $B_n + 3B_{n-1} + B_{n-2} = 0$

$B_n = -3B_{n-1} - B_{n-2}$

Comment:

The results from part (i) needed to be used in some way.

(a) (continued)

(iii) Hence, briefly explain why $B_n \in \mathbb{Z}$ for all positive integers n. **3**

Method 1:

$$B_1 = \beta + \frac{1}{\beta}$$
$$= \frac{-3+\sqrt{5}}{2} + \frac{2}{-3+\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$
$$= \frac{-3+\sqrt{5}}{2} + \frac{6+2\sqrt{5}}{-4}$$
$$= \frac{-3+\sqrt{5}}{2} + \frac{-3-\sqrt{5}}{2}$$
$$= -3 \in \mathbb{Z}.$$

$$B_2 = \beta^2 + \frac{1}{\beta^2}$$
$$= \left(\beta + \frac{1}{\beta}\right)^2 - 2$$
$$= (-3)^2 - 2 \qquad (\because B_1 = -3 \text{ (from above)})$$
$$= 7 \in \mathbb{Z}$$

$\therefore B_1, B_2 \in \mathbb{Z}$

From (ii), $B_3 = -3B_2 - B_1 \in \mathbb{Z}$, since $\mathbb{Z}$ is closed under multiplication and subtraction.

Similarly, $B_4 \in \mathbb{Z}$, $B_5 \in \mathbb{Z}$, ... and so on.

$\therefore B_n \in \mathbb{Z}$ for all positive integers n

Method 2: Induction

Over the page

(a) (iii) (continued)

Prove true for $n=1$

$B_1 = \beta^1 + \beta^{-1}$ (sum of roots of $x^2+3x+1=0$)

$= -\frac{b}{a}$

$= -\frac{3}{1} \in \mathbb{Z}$

$\therefore$ true for $n=1$

Prove true for $n=2$

$B_2 = \beta^2 + \beta^{-2}$

$= (\beta^1 + \beta^{-1})^2 - 2\beta\beta^{-1}$

$= (-3)^2 - 2(1)$

$= 7 \in \mathbb{Z}$

$\therefore$ true for $n=2$

Assume true for $n=k, k+1$, $k \in \mathbb{Z}^+$

ie $B_k , B_{k+1} \in \mathbb{Z}$

Prove true for $n=k+2$

$B_{k+2} = -3B_{k+1} - B_k$

since $B_k, B_{k+1} \in \mathbb{Z}$

and integers are closed under subtraction and scalar multiplication.

$B_{k+2} \in \mathbb{Z}$

$\therefore$ true for $n=k+2$

By the principle of mathematical induction $B_n \in \mathbb{Z}$ for all positive integers n.

(a) (iii) (continued)

Comment:

$B_0 = \beta^0 + \beta^{-0}$

$= 2$

could have been used along with B_1 to prove $B_n \in \mathbb{Z}$ for integers $n \in \{0, 1, 2, 3, \ldots\}$.

This would mean $B_n \in \mathbb{Z}$ for all positive integers n.

(b) Consider the polynomial $P(z) = z^{2n+1} - 1$.

(i) Prove that the roots of $P(z)$ are $z = \alpha_k = \cos\theta_k + i\sin\theta_k$ where $\theta_k = \dfrac{2k\pi}{2n+1}$ for $k = 0, 1, 2, \ldots, 2n$. **1**

$P(z) = 0$ for $n \in \mathbb{Z}^+$

$z^{2n+1} - 1 = 0$

$z^{2n+1} = 1$

let $z = r(\cos\theta + i\sin\theta)$

$(r(\cos\theta + i\sin\theta))^{2n+1} = 1(\cos 2k\pi + i\sin 2k\pi)$ for $k \in \mathbb{Z}$

$r^{2n+1}(\cos(2n+1)\theta + i\sin(2n+1)\theta) = 1(\cos 2k\pi + i\sin 2k\pi)$

equate

$r^{2n+1} = 1$ $\quad (2n+1)\theta = 2k\pi$

$\therefore r = 1$ $\quad \theta = \dfrac{2k\pi}{2n+1}$

Since $P(z)$ is a polynomial of degree $(2n+1)$ [for $n \in \mathbb{Z}^+$]
it will have $(2n+1)$ complex zeros.

ie $k = 0, 1, 2, \ldots, 2n$.

$\therefore$ zeros of $P(z)$ are $z = \alpha_k = \cos\theta_k + i\sin\theta_k$

where $\theta_k = \dfrac{2k\pi}{2n+1}$ for $k = 0, 1, 2, \ldots, 2n$.

Comment:

Students should not skip any lines of working for a proof.

Also De Moivre's Theorem is for integer powers.

(b) (continued)

(ii) Prove that $\overline{\alpha_k} = \alpha_{2n+1-k}$. **1**

$$\overline{\alpha_k} = \overline{\left(\cos\frac{2k\pi}{2n+1} + i\sin\frac{2k\pi}{2n+1}\right)}$$

$$= \cos\frac{2k\pi}{2n+1} - i\sin\frac{2k\pi}{2n+1}$$

$$= \cos\left(2\pi - \frac{2k\pi}{2n+1}\right) + i\sin\left(2\pi - \frac{2k\pi}{2n+1}\right)$$

$$= \cos\left(\frac{2\pi(2n+1) - 2k\pi}{2n+1}\right) + i\sin\left(\frac{2\pi(2n+1) - 2k\pi}{2n+1}\right)$$

$$= \cos\left(\frac{2\pi(2n+1-k)}{2n+1}\right) + i\sin\left(\frac{2\pi(2n+1-k)}{2n+1}\right)$$

$$= \alpha_{2n+1-k}$$

Comment

This is a proof. Students can't just state that the zeros will be in conjugate pairs. They need to match the specific values of k.

(b) (continued)

(iii) It is known that **2**

$$\prod_{k=1}^{n} T_k = T_1 \times T_2 \times T_3 \times \cdots \times T_{n-1} \times T_n.$$

Hence, or otherwise, prove that

$$z^{2n} + z^{2n-1} + \cdots + z + 1 = \prod_{k=1}^{n} (z - \alpha_k)(z - \overline{\alpha_k}).$$

$P(z) = z^{2n+1} - 1$

$= (z-\alpha_0)(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{2n-1})(z-\alpha_{2n})$

$= (z-1)(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{2n+1-2})(z-\alpha_{2n+1-1})$

$= (z-1)(z-\alpha_1)(z-\alpha_{2n+1-1})(z-\alpha_2)(z-\alpha_{2n+1-2})\ldots(z-\alpha_n)(z-\alpha_{2n+1-n})$

$= (z-1)(z-\alpha_1)(z-\overline{\alpha_1})(z-\alpha_2)(z-\overline{\alpha_2})\ldots(z-\alpha_n)(z-\overline{\alpha_n})$

Also, $z^{2n+1} - 1$

$= (z-1)(z^{2n} + z^{2n-1} + z^{2n-2} + \ldots + z^2 + z + 1)$

equate these two expressions

$z^{2n} + z^{2n-1} + \ldots + z + 1 = (z-\alpha_1)(z-\overline{\alpha_1})(z-\alpha_2)(z-\overline{\alpha_2})\ldots(z-\alpha_n)(z-\overline{\alpha_n})$

$= \prod_{k=1}^{n} (z-\alpha_k)(z-\overline{\alpha_k})$

Comment

This is a proof. Students need to connect the linear pairs.

Note: $z=1$ is a complex zero. It is best to describe the other zeros as non-real zeros as opposed complex zeros.

(b) (continued)

(iv) Let $A_k = 5 - 4\sin^2\left(\dfrac{k\pi}{2n+1}\right)$. **2**

Show that $A_k = (\alpha_k - \beta)\left(1 - \alpha_k^{-1}\beta^{-1}\right)$, where β is the value from (a).

$$
\begin{aligned}
A_k &= (\alpha_k - \beta)(1 - \alpha_k^{-1}\beta^{-1}) \\
&= (\alpha_k - \beta)(1 - \overline{\alpha_k}\,\beta^{-1}) \qquad \text{since } |\alpha_k| = 1,\ \tfrac{1}{\alpha_k} = \overline{\alpha^k} \\
&= \alpha_k - \alpha_k\overline{\alpha}_k\beta^{-1} - \beta + \overline{\alpha}_k\beta\beta^{-1} \\
&= \alpha_k - \beta^{-1} - \beta + \overline{\alpha}_k \\
&= \alpha_k + \overline{\alpha}_k - (\beta + \beta^{-1}) \\
&= 2\,\mathrm{Re}(\alpha_k) - (-3) \qquad \text{from (a)} \\
&= 3 + 2\cos\left(\frac{2k\pi}{2n+1}\right) \\
&= 3 + 2\left[1 - 2\sin^2\left(\frac{k\pi}{2n+1}\right)\right] \\
&= 3 + 2 - 4\sin^2\left(\frac{k\pi}{2n+1}\right) \\
&= 5 - 4\sin^2\left(\frac{k\pi}{2n+1}\right) \\
&= A_k
\end{aligned}
$$

Comment

Working should be clear to follow as multiple connections are required.

(b) (continued)

(v) Hence, or otherwise, prove that $S = \prod_{k=1}^{n} \left[5 - 4\sin^2\left(\frac{k\pi}{2n+1}\right)\right]$ is an integer. **3**

You may assume the result from Question 16 (a) (iii).

$$S = \prod_{k=1}^{n} \left[5 - 4\sin^2\left(\frac{k\pi}{2n+1}\right)\right]$$

$$= \prod_{k=1}^{n} A_k$$

$$= \prod_{k=1}^{n} (\alpha_k - \beta)(1 - \alpha_k^{-1}\beta^{-1})$$

$$= \prod_{k=1}^{n} (\alpha_k - \beta)\left(1 - \frac{\bar{\alpha}_k}{\beta}\right)$$

$$= \prod_{k=1}^{n} \frac{1}{\beta}(\alpha_k - \beta)(\beta - \bar{\alpha}_k)$$

$$= \prod_{k=1}^{n} -\frac{1}{\beta}(\beta - \alpha_k)(\beta - \bar{\alpha}_k)$$

$$= \frac{(-1)^n}{\beta^n} \prod_{k=1}^{n} (\beta - \alpha_k)(\beta - \bar{\alpha}_k)$$

[let $z = \beta$ in (iii)

ie $\beta^{2n} + \beta^{2n-1} + \ldots + \beta + 1 = \prod_{k=1}^{n} (\beta - \alpha_k)(\beta - \bar{\alpha}_k)$]

$$= \frac{(-1)^n}{\beta^n}\left[\beta^{2n} + \beta^{2n-1} + \ldots + \beta^{n+1} + \beta^n + \beta^{n-1} + \ldots + \beta + 1\right]$$

$$= (-1)^n\left[\beta^n + \beta^{n-1} + \ldots + \beta + 1 + \beta^{-1} + \ldots + \beta^{-n+1} + \beta^{-n}\right]$$

$$= (-1)^n\left[\beta^n + \beta^{-n} + \beta^{n-1} + \beta^{-n+1} + \ldots + \beta + \beta^{-1} + 1\right]$$

$$= (-1)^n\left[B_n + B_{n-1} + \ldots + B_1 + 1\right]$$

since $B_n \in \mathbb{Z}$ (from (a))

$S \in \mathbb{Z}$

(b) (v) (continued)

Comment

Very few students were able to get this. A lot of connections with previous parts needed to be made.

End of Q16 Solutions

Section I Multiple Choice Solutions

1 Which is the correct expression for $\int \frac{3x+1}{x^2+2x+4}dx$?

A. $\frac{3}{2}\ln(x^2+2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

B. $3\ln(x^2+2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

C. $\frac{3}{2}\ln(x^2+2x+4) - 2\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

D. $3\ln(x^2+2x+4) - 2\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

Because of the $\frac{3}{2}$ then options A and C

As $x^2 + 2x + 4 = (x + 1)^2 + 3$ then options A or B

More formal solution:

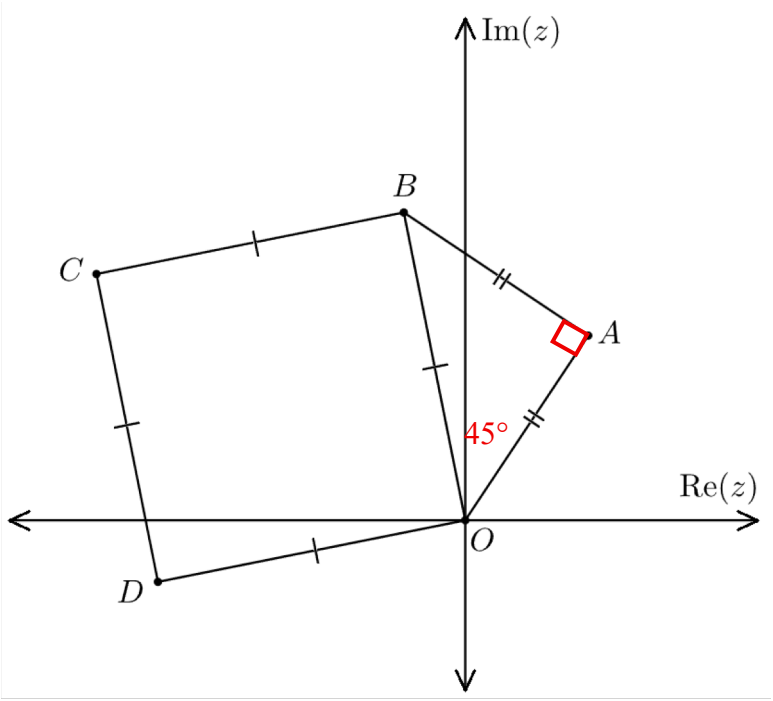
$$\int \frac{3x+1}{x^2+2x+4}\,dx = \int \frac{\frac{3}{2}(2x+2) - 2}{x^2+2x+4}\,dx$$

$$= \frac{3}{2}\int \frac{2x+2}{x^2+2x+4}\,dx - 2\int \frac{1}{(x+1)^2 + (\sqrt{3})^2}\,dx$$

$$= \frac{3}{2}\ln\left(x^2+2x+4\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

A	115
B	0
C	3
D	0

2 In the diagram, $\triangle OAB$ is a right-angled isosceles triangle and $OBCD$ is a square. Let A represent the complex number α and D represent the complex number β.



Which of the following is correct?

A. $\beta + \alpha = i\alpha$

B. $\beta - \alpha = i\alpha$

C. $\sqrt{2}\beta + \alpha = i\alpha$

D. $\sqrt{2}\beta - \alpha = i\alpha$

$$\begin{aligned}\overrightarrow{OB} &= \sqrt{2}\,\overrightarrow{OA}\,e^{\frac{i\pi}{4}} \\ &= (1 + i)\,\alpha \\ \overrightarrow{OD} &= i\,\overrightarrow{OB} \\ \therefore \beta &= i\,(1 + i)\,\alpha \\ &= i\alpha - \alpha\end{aligned}$$

Options C & D are obviously wrong as the scale factor, $\sqrt{2}$ has been applied to the wrong vector.

A	90
B	4
C	19
D	5

3 A particle moving in simple harmonic motion centred at the origin attains minimum acceleration of $-3\ \text{m/s}^2$ and a maximum speed of $4\ \text{m/s}$. At what position does it attain three quarters of its maximum acceleration?

A. $x = \frac{16}{3}$

B. $x = -\frac{16}{3}$

C. $x = 4$

D. $x = -4$

Maximum acceleration is when $x < 0$
So only options B and D.
Need the one with smaller magnitude.

Let the amplitude be A.

$a = -n^2x \Rightarrow -3 = -n^2A$ [Minimum acceleration occurs at maximum displacement]

$\therefore n^2A = 3$

Maximum speed occurs at $x = 0$
As $v^2 = n^2(A^2 - x^2)$ then $16 = n^2A^2 \Rightarrow A = \frac{16}{3}$

A	12
B	37
C	25
D	44

4 Let u be a complex number with $|u| \leq 5$. What is the minimum value of $|2u + 3 + 4i|$?

A. -5

B. 0

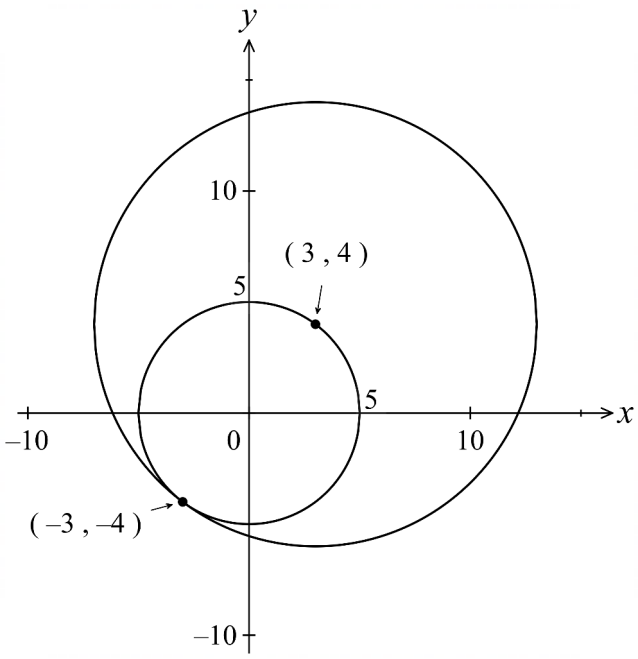
C. 10

D. 15

Options A is just wrong and D is for "maximum".

If $u = -3 - 4i$ then $|2u + 3 + 4i| = 0$

Graphically: The locus of $2u + 3 + 4i$ is a circle radius 10 centred at (3, 4),
i.e. the locus of u dilated by a factor of 2 and translated to the right 2 units and up 4 units



Triangle Inequality: The variant is $\big||a| - |b|\big| \leq |a - b|$.

$$\begin{aligned} |2u + 3 + 4i| &= |2u - (-3 - 4i)| \\ &\geq \big||2u| - |-3 - 4i|\big| \\ &= |10 - 5| \\ &= 5 \end{aligned}$$

A	7
B	75
C	25
D	11

5 Let $\text{Arg}\,(z)$ be the principal argument of z. How many possible real values of k can satisfy

$$\text{Arg}\,(zw) = \text{Arg}\,(z) + \text{Arg}\,(w) + k\pi$$

for complex numbers z and w?

A. 2

B. 3

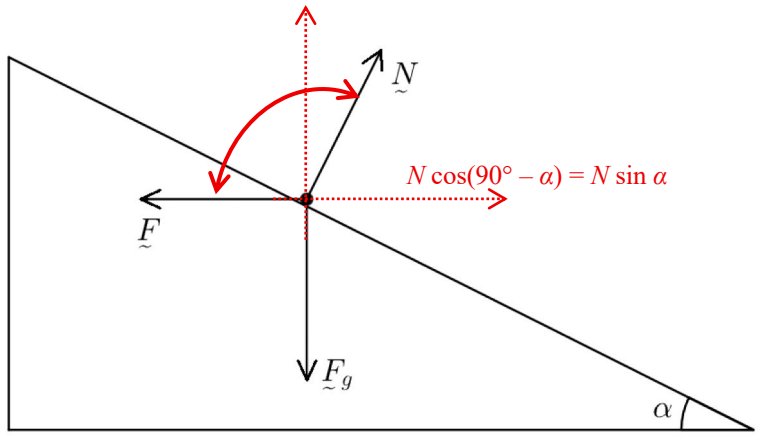
C. 4

D. 5

As $-\pi < \text{Arg}\ z \leq \pi$ then $-2\pi < \text{Arg}\ z + \text{Arg}\ w \leq 2\pi$

A	39
B	44
C	26
D	9

6 A ball is held stationary on a slope by a horizontal force $\underset{\sim}{F}$ and is subject to gravity $\underset{\sim}{F}_g$ and the normal force $\underset{\sim}{N}$ as shown.



Which of the following is equal to $\underset{\sim}{F} \cdot \underset{\sim}{N}$?

A. $|\underset{\sim}{F}||\underset{\sim}{F}_g|$

B. $|\underset{\sim}{F}|^2$

C. $-|\underset{\sim}{F}||\underset{\sim}{F}_g|$

D. $-|\underset{\sim}{F}|^2$

Resolving horizontally: As the ball is stationary $\underset{\sim}{F} + \underset{\sim}{N}\sin\alpha = 0$

$$\underset{\sim}{F} \cdot \underset{\sim}{N} = |\underset{\sim}{F}|\,|\underset{\sim}{N}|\cos(90° + \alpha)$$

$$= |\underset{\sim}{F}| \times (-|\underset{\sim}{N}|\sin\alpha)$$

$$= -|\underset{\sim}{F}|^2$$

A	9
B	14
C	56
D	39

7 Let $P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_0$ be a polynomial with integer coefficients, where n is an even positive integer.

$P(z)$ is known to have zeros at $z = 2 + \sqrt{3}$, $z = -\sqrt{3}$ and $z = -4 + 2i$.

What is the minimum possible degree of the $P(z)$?

A. 6

B. 8

C. 10

D. 12

As the coefficients are real then $P\left(-4 \pm 2i\right) = 0$

As the coefficients are integers then $P\left(2 \pm \sqrt{3}\right) = 0$

As $P(z)$ is even: if $P(\alpha) = 0$ then $P(-\alpha) = 0$

A	62
B	46
C	9
D	1

8 f is an even continuous function. Which of the following is not always equal to $\int_0^{2a} f(x)dx$?

A. $\int_0^a \left[f(x)+f(2a-x)\right]dx$

B. $\int_{-a}^0 \left[f(x)+f(a-x)\right]dx$

C. $\int_{-a}^0 \left[f(x)+f(-2a-x)\right]dx$

D. All of the above are equal.

Note: $\int_0^{2a} = \int_0^a + \int_a^{2a}$

A. False $\int_0^a f(2a-x)\,dx = \int_a^{2a} f(u)\,du$, for $u = 2a - x$

As f is even then $\int_{-a}^0 f(x)\,dx = \int_0^a f(x)\,dx$

B. False $\int_{-a}^0 f(a-x)\,dx = \int_a^{2a} f(u)\,du$, for $u = a - x$

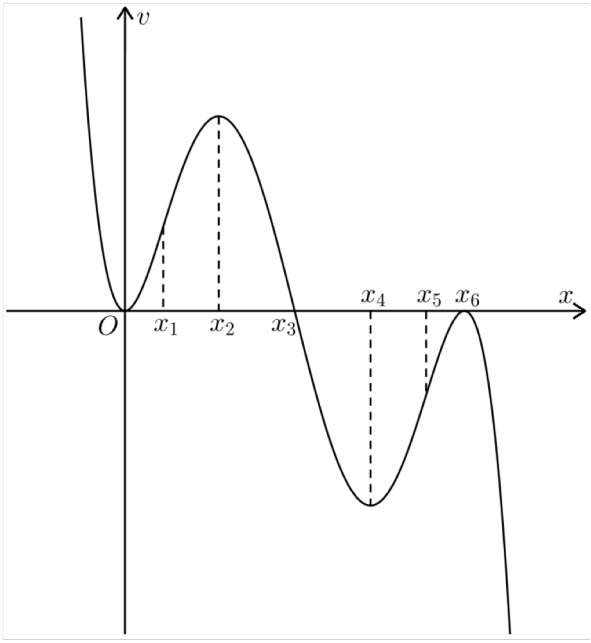
C. False $f(-2a-x) = f(2a+x)$

$\int_{-a}^0 f(2a+x)\,dx = \int_a^{2a} f(u)\,dx$, for $u = 2a + x$

∴ They are all equal to $\int_0^{2a} f(x)\,dx$.

A	12
B	43
C	33
D	30

9 A particle is initially placed at a random position on the x-axis. The particle's velocity v as a function of x is shown in the diagram below.



At what range of initial positions will it experience positive acceleration?

A. $0 < x < x_2$ or $x_4 < x < x_6$

B. $x < x_1$ or $x_3 < x < x_5$

C. $0 < x < x_2$ or $x_3 < x < x_4$ or $x > x_6$

D. $0 < x < x_2$ or $x_3 < x < x_6$

As $a = v\dfrac{dv}{dx}$, we need where $v > 0$ AND $\dfrac{dv}{dx} > 0$ **OR** $v < 0$ AND $\dfrac{dv}{dx} < 0$

A	56
B	12
C	47
D	3

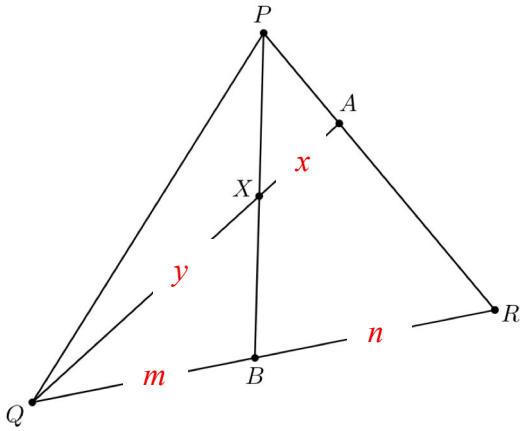
Possibly students thought the x-axis was a t-axis.
If students had looked at the 2025 HSC ME 2 paper, they would have seen a question with these axes.

For revision: look at the NSGHS 2026 ME 2 THSC q10.

10 The diagram shows triangle $\triangle PQR$.

A is on PR such that $PA : PR = 1 : 4$.

B is on QR such that $QB : BR = m : n$ where m, n are odd positive integers.



If the point W divides an interval UV so that $UW : WV = a : b$ and the position vectors of U and V are $\underset{\sim}{u}$ and $\underset{\sim}{v}$ respectively, then $\overrightarrow{OW} = \dfrac{a}{a+b}\underset{\sim}{v} + \dfrac{b}{a+b}\underset{\sim}{u}$.

Let X be on AQ such that $AX : XQ = x : y$ where x, y are positive integers that add to 47. By finding two different expressions for $\overrightarrow{PX}$, what is the minimum value of x?

A. 2

B. 3

C. 4

D. 5

A	10
B	39
C	40
D	29

10 (continued)

Let $\overrightarrow{PQ} = \underset{\sim}{q}$ and $\overrightarrow{PR} = \underset{\sim}{r}$

From the information given in the question: $\overrightarrow{PB} = \frac{n}{m+n}\underset{\sim}{q} + \frac{m}{m+n}\underset{\sim}{r}$

1. $\overrightarrow{PX} = \mu\,\overrightarrow{PB}$, where $\mu \in \mathbb{R}$.

$$\begin{aligned}\overrightarrow{AQ} &= \overrightarrow{AP} + \overrightarrow{PQ} \\ &= -\frac{1}{5}\underset{\sim}{r} + \underset{\sim}{q}\end{aligned}$$

2. $$\begin{aligned}\overrightarrow{PX} &= \overrightarrow{PA} + \overrightarrow{AX} \\ &= \frac{1}{5}\overrightarrow{PR} + \frac{x}{47}\overrightarrow{AQ} \\ &= \frac{1}{5}\underset{\sim}{r} + \frac{x}{47}\left(-\frac{1}{5}\underset{\sim}{r} + \underset{\sim}{q}\right) \\ &= \frac{1}{5}\left(1 - \frac{x}{47}\right)\underset{\sim}{r} + \frac{x}{47}\underset{\sim}{q}\end{aligned}$$

Equating coefficients of $\underset{\sim}{r}$ and $\underset{\sim}{q}$:

$$\mu\left(\frac{n}{m+n}\right) = \frac{1}{5}\left(1 - \frac{x}{47}\right) = \frac{47 - x}{235} \qquad -(1)$$

$$\mu\left(\frac{m}{m+n}\right) = \frac{x}{47} \qquad -(2)$$

(1) ÷ (2):

$$\frac{n}{m} = \frac{5(47 - x)}{x}$$

$\because$ m and n are both odd $\Rightarrow x = 3$ is smallest value of x

Note: even though HCF(m , n) might not be 1, $\frac{n}{m} \neq \frac{c}{2}$.