Sydney Girls High School
2026
Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hour and 45 minutes for this section

Name: …… Teacher: ……	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2026 HSC Examination Paper in this subject.

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let a and b be real numbers.
Which of the following is equal to $(a + ib)^{2026}$?

A. $(b - ia)^{2026}$

B. $-(b - ia)^{2026}$

C. $i(b - ia)^{2026}$

D. $-i(b - ia)^{2026}$

2 It is given that $2 + 5i$ is a root of $P(z) = z^3 - 8z^2 + 45z - 116$.
Which of the following is a quadratic factor of $P(z)$?

A. $z^2 - 4z + 29$

B. $z^2 + 4z + 29$

C. $z^2 - 4z - 29$

D. $z^2 + 4z - 29$

3 For which of the following integers m is $(\sqrt{3} + i)^m$ real?

A. 3

B. 4

C. 5

D. 6

4 Which of the following is equal to $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\, dx$?

A. $\sqrt{2}$

B. $\dfrac{1}{2}$

C. $\ln\left(1+\sqrt{2}\right)$

D. $\ln\dfrac{\pi}{4}$

5 Let $P(8,-4,12)$ and $Q(-12,0,8)$ be points in three-dimensional space. Which of the following points divides PQ internally in the ratio $1:3$?

A. $(-3,-3,11)$

B. $(3,-3,11)$

C. $(-1,-2,8)$

D. $(1,-2,8)$

6 Which of the following is equal to $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x + \tan^2 x}\, dx$?

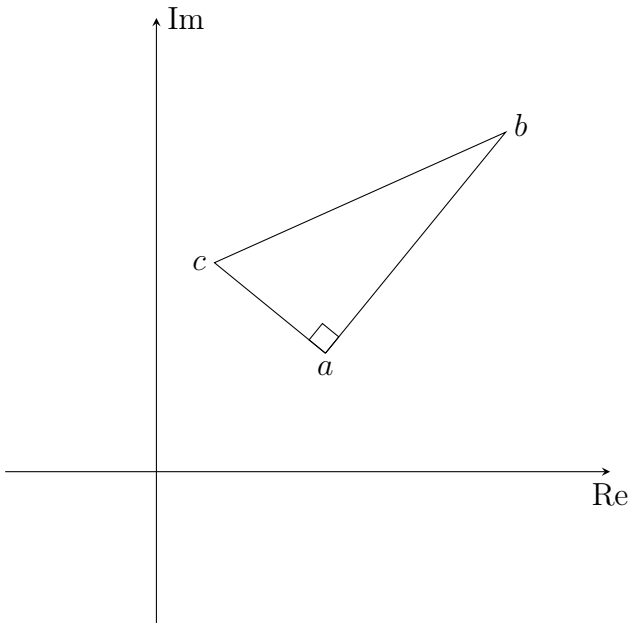
A. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{t(t+2)}\, dt$

B. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{t(t+1)}\, dt$

C. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{t^2(t^2+2)}\, dt$

D. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{t^2(t^2+1)}\, dt$

7 The complex numbers a, b and c form a right-angled triangle with right-angle at a when plotted on the Argand diagram, as shown below.



Which of the following is a correct expression for c?

A. $a + i\left|\dfrac{a-c}{a-b}\right|(b-a)$

B. $a - i\left|\dfrac{a-c}{a-b}\right|(b-a)$

C. $a + i\left|\dfrac{a-b}{a-c}\right|(b-a)$

D. $a - i\left|\dfrac{a-b}{a-c}\right|(b-a)$

8 Let $k > 0$ be a real number and suppose $\text{Arg}(z+k) = \dfrac{\pi}{6}$ and $\text{Arg}(z-k) = \dfrac{2\pi}{3}$. Which of the following is equal to $|z|$?

A. k

B. $2k$

C. $3k$

D. $4k$

9 Which of the following statements is true?

A. $\forall r \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ such that $0 \leqslant r - n < 1$

B. $\exists n \in \mathbb{Z}$ such that $\forall r \in \mathbb{R}, 0 \leqslant r - n < 1$

C. $\forall r \in \mathbb{R}^+$, $\exists n \in \mathbb{Z}^+$ such that $0 \leqslant r - n < 1$

D. $\exists n \in \mathbb{Z}^+$ such that $\forall r \in \mathbb{R}^+, 0 \leqslant r - n < 1$

10 Consider the complex number

$$z = \left(1 + \frac{i}{\sqrt{1}}\right)\left(1 + \frac{i}{\sqrt{2}}\right)\cdots\left(1 + \frac{i}{\sqrt{2026}}\right).$$

The argument of each factor of z can be estimated using the small angle approximation $\arctan x \approx x$. After these estimates have been made, $\arg(z)$ can be approximated using lower rectangles under the curve $y = \dfrac{1}{\sqrt{x}}$.
According to this method, which of the following is the best approximation of $\mathrm{Re}(z)$?

A. $\sqrt{2026}\cos\left(2\sqrt{2026}\right)$

B. $\sqrt{2026}\cos\left(2\sqrt{2027}\right)$

C. $\sqrt{2027}\cos\left(2\sqrt{2026}\right)$

D. $\sqrt{2027}\cos\left(2\sqrt{2027}\right)$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Start each question on a new page.

Question 11 (15 marks) Start a new page.

(a) Express $(4+i)\overline{(3-2i)}$ in the form $x+iy$, where x and y are real. **1**

(b) Solve $4z^2-4z+17=0$. Give your answers in simplest Cartesian form. **2**

(c) Express $-2\sqrt{3}-2i$ in the form $re^{i\theta}$, where r and θ are real and $r>0$. **2**

(d) Evaluate $\displaystyle\int_0^1 \frac{1-x^2}{1+x^2}\,dx$. **2**

(e) Using a suitable substitution, or otherwise, find $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$. **2**

(f) Find the vector with magnitude 12 and the same direction as $-\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$. **2**

(g) Sketch the set of all z satisfying $|z+3-2i|=|z+1+2i|$. **2**
Label the intercepts with the coordinate axes.

(h) It is given that $2-i$ is one of the zeroes of $P(z)=z^3-12z^2+37z-40$. **2**
Find the other two zeroes, giving reasons for your answer.

Question 12 (15 marks) Start a new page.

(a) Show that ${}^nP_r$ is divisible by r. **2**

(b) Use the identities $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ and $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ to prove that $1 + \tan^2\theta = \sec^2\theta$. **2**

(c) Find the area of the quadrilateral formed by $z, -z, 2iz$ and $-2iz$ if $|z| = 4$. **2**

(d) Evaluate $\displaystyle\int_{-6}^{3} \frac{1}{\sqrt{45 - 12x - x^2}}\, dx$. **2**

(e) Let $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. **2**

It is given that $|2\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{a} - 2\underset{\sim}{b}| = \sqrt{10}$.

Calculate the exact angle between $2\underset{\sim}{a} - \underset{\sim}{b}$ and $\underset{\sim}{a} - 2\underset{\sim}{b}$.

(f) A particle of mass 6 kg moves in a straight line subject to a force $F = \dfrac{1}{v+3}$ newtons. At time t seconds, the displacement of the particle is x metres and the velocity is v ms^{-1}. **2**

The particle was initially at rest at the origin.

Find x when $v = 4$ ms^{-1}.

(g) Find $\displaystyle\int \frac{8x + 2}{(3x - 2)(x^2 + 2)}\, dx$. **3**

Question 13 (15 marks) Start a new page.

(a) By writing $1-i$ and $1-i\sqrt{3}$ in mod-arg form, find the exact value of $\sin\dfrac{7\pi}{12}$. **3**

(b) It is given that if x^2 divides y^2 then x divides y for all integers x and y. **3**
Suppose that n is a positive integer and that $\sqrt{n}$ is not an integer.
Prove by contradiction that $\sqrt{n}$ is irrational.

(c) Evaluate $\displaystyle\int_1^e \cos(\ln x)\,dx$. **3**

(d) The points A, B and P have position vectors $\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + \lambda\underset{\sim}{k}$, **3**
$\overrightarrow{OB} = 5\underset{\sim}{i} + \lambda\underset{\sim}{j}$ and $\overrightarrow{OP} = -\underset{\sim}{i} - \lambda\underset{\sim}{j} + \underset{\sim}{k}$ respectively.
Find the possible values of λ if $\angle AOP = \angle BOP$.

(e) A particle is moving in simple harmonic motion with a period of 6π seconds **3**
according to the equation $x = a\cos(nt + \alpha)$, where x is the displacement in metres, $a > 0$, and $0 \leqslant \alpha < 2\pi$.

When $t = 2$, the particle's displacement is $x = -2$ metres and its velocity is $v = 1$ metre per second.

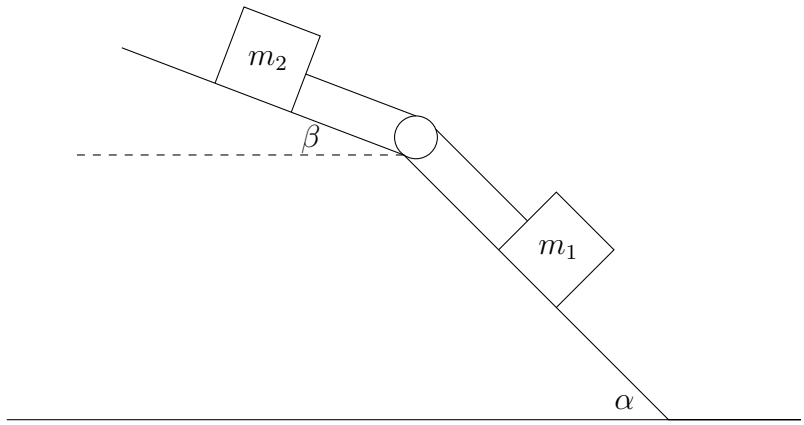
Find a and α correct to four significant figures.

Question 14 (15 marks) Start a new page.

(a) Use mathematical induction to prove that $\dfrac{1}{1^3}+\dfrac{1}{2^3}+\dfrac{1}{3^3}+\cdots+\dfrac{1}{n^3} \leqslant \dfrac{3}{2}-\dfrac{1}{2n^2}$ for all positive integers n. **3**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x \sec^6 x \, dx$. **3**

(c) Two boxes of masses m_1 kg and m_2 kg are connected by a light inextensible string running over a frictionless pulley. The tension in the string is T newtons. The m_1 kg box slides down a frictionless plane at an angle of α to the horizontal, and the m_2 kg box slides down a frictionless plane at an angle of β to the horizontal, as shown below. **3**



Show that the acceleration of the connected body is

$$a = \frac{g}{m_1 + m_2}\Big(m_1 \sin\alpha + m_2 \sin\beta\Big).$$

(d) (i) Use De Moivre's Theorem to prove that **3**

$$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7.$$

You may let $c = \cos\theta$ and $s = \sin\theta$ to simplify your working.

(ii) Hence, by using a suitable substitution, solve $x^3 - 7x^2 + 14x - 7 = 0$. **2**

(iii) Without using a calculator, evaluate $\cos^4\dfrac{\pi}{14} + \cos^4\dfrac{3\pi}{14} + \cos^4\dfrac{5\pi}{14}$. **1**

Question 15 (15 marks) Start a new page.

(a) A particle moves in simple harmonic motion according to the equation $v^2 = 8 - 10x - 3x^2$. Find the centre of motion and the amplitude. **2**

(b) (i) Let n be an odd positive integer, and let $\omega = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$. **2**

Show that $1 - \omega + \omega^2 - \cdots + (-1)^{k-1}\omega^{k-1} + \cdots + \omega^{n-1} = 1 - i\tan\dfrac{\pi}{n}$.

(ii) Deduce that $1 + \omega^2 + \omega^4 + \cdots + \omega^{n-1} = \dfrac{1}{2}\left(1 - i\tan\dfrac{\pi}{n}\right)$. **1**

(c) Let $I_n = \displaystyle\int_0^r x^{n+\frac{1}{2}}\sqrt{r-x}\,dx$, where $n \geqslant 0$ is an integer and $r > 0$ is real. **3**

Show that $I_n = \dfrac{(2n+1)r}{2n+4} I_{n-1}$ for $n \geqslant 1$.

(d) (i) Suppose that the line $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ intersects the sphere $|\underset{\sim}{v} - \underset{\sim}{c}| = R$. **2**

Show that $|\underset{\sim}{b}|^2\lambda^2 + 2\lambda\,\underset{\sim}{b}\cdot(\underset{\sim}{a} - \underset{\sim}{c}) + |\underset{\sim}{a} - \underset{\sim}{c}|^2 - R^2 = 0$.

(ii) Deduce that if R is the shortest distance between the line and the centre of the sphere then $R^2 = |\underset{\sim}{a} - \underset{\sim}{c}|^2 - \left(\dfrac{\underset{\sim}{b}\cdot(\underset{\sim}{a} - \underset{\sim}{c})}{|\underset{\sim}{b}|}\right)^2$. **2**

(iii) Hence, or otherwise, show that $R = |\underset{\sim}{a} - \underset{\sim}{c}|\sin\theta$, where θ is the angle between $\underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{c}$. **1**

(iv) Use (ii) to find the shortest distance between the lines **2**

$$\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix}$$

Question 16 (15 marks) Start a new page.

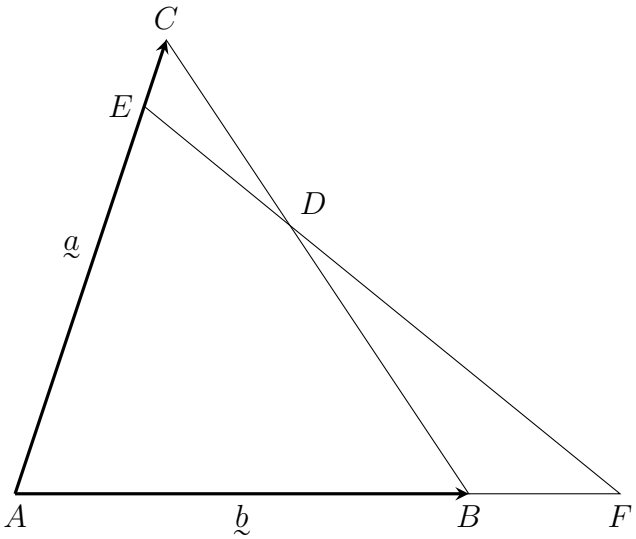
(a) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geqslant 0$, where $a_0, a_1, \ldots, a_n$ are non-negative integers.

Suppose that $P(x)$ is equal to a prime number for every integer input x.

(i) Show that $P(a_0) = a_0$. **3**

(ii) Deduce that $P(x) = a_0$ for all real x. **1**

(b) In the diagram below, the triangle $\triangle ABC$ is intersected by the line EDF. **4**
The point F lies on the line AB extended.



Let $\overrightarrow{AC} = \underset{\sim}{a}$ and $\overrightarrow{AB} = \underset{\sim}{b}$.

Using vectors, show that $\dfrac{\left|\overrightarrow{AF}\right|}{\left|\overrightarrow{BF}\right|} \times \dfrac{\left|\overrightarrow{DB}\right|}{\left|\overrightarrow{CD}\right|} \times \dfrac{\left|\overrightarrow{EC}\right|}{\left|\overrightarrow{AE}\right|} = 1.$

Question 16 continues on the next page

(c) Let n be a positive integer.

It is given that for non-negative real numbers $x_1, x_2, x_3 \ldots, x_n$,

$$\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)^n \geqslant x_1 x_2 x_3 \ldots x_n. \quad \text{(Do not prove this.)}$$

(i) Show that $\dfrac{n^n}{n!} \geqslant 2^{n-1}$. **3**

(ii) Let $\lfloor x \rfloor$ be greatest integer that is less than or equal to x. **2**

For example, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 2.718 \rfloor = 2$.

Let $f(x) = \dfrac{x^x}{\lfloor x \rfloor!}$ for $x \geqslant 1$.

Show that $f(x)$ is increasing for $x \in [a, a+1)$, where $a \geqslant 1$ is an integer.

(iii) Hence, show that $\displaystyle\int_1^{n+1} \frac{x^x}{\lfloor x \rfloor!}\, dx \geqslant 2^n - 1$. **2**

End of paper

Sydney Girls High School
2026 Extension 2 Mathematics Trial
Marking Guidelines

Multiple choice Answer Key

Question	Answer
1	B
2	A
3	D
4	C
5	B
6	C
7	A
8	A
9	A
10	C

Question 1	1 mark

Solution

First note that $a+ib = i(b-ia)$.

Therefore

$$\begin{aligned}(a+ib)^{2026} &= i^{2026}(b-ia)^{2026}\\ &= i^2 \times (i^4)^{506} \times (b-ia)^{2026}\\ &= -1 \times 1^{506} \times (b-ia)^{2026}\\ &= -(b-ia)^{2026}\end{aligned}$$

Hence (B). ✓

Question 2	1 mark

Solution

All coefficients are real, so the conjugate root theorem implies that $2-5i$ is a root, since $2+5i$ is a root. Therefore the required quadratic factor is

$$\begin{aligned}(z-\alpha)(z-\overline{\alpha}) &= z^2 - 2\text{Re}(\alpha)z + |\alpha|^2\\ &= z^2 - 2\text{Re}(2+5i)z + |2+5i|^2\\ &= z^2 - 4z + 29\end{aligned}$$

Hence (A). ✓

Question 3	1 mark

Solution

$$\begin{aligned}(\sqrt{3}+i)^m &= \left(2\text{cis}\frac{\pi}{6}\right)^m\\ &= 2^m\text{cis}\frac{m\pi}{6}\end{aligned}$$

This is real provided that

$$\begin{aligned}\sin\frac{m\pi}{6} &= 0\\ \frac{m\pi}{6} &= 0, \pm\pi, \pm 2\pi, \ldots\\ m &= 0, \pm 6, \pm 12, \ldots\end{aligned}$$

Hence (D). ✓

Question 4	**1 mark**

Solution

Note that $\dfrac{d}{dx}(\tan x + \sec x) = \sec^2 x + \sec x \tan x$. This means that the integral is of the form $\displaystyle\int_0^{\frac{\pi}{4}} \frac{f'(x)}{f(x)}\,dx = [\ln(|f(x)|)]_0^{\frac{\pi}{4}}$, where $f(x) = \tan x + \sec x$.
Hence, the answer is

$$\begin{aligned}[\ln|\tan x + \sec x|]_0^{\frac{\pi}{4}} &= \ln(1+\sqrt{2}) - \ln 1 \\ &= \ln(1+\sqrt{2})\end{aligned}$$

Hence (C). ✓

Question 5	**1 mark**

Solution
We will first find the midpoint, M, of PQ. Then we will find the midpoint, N, of PM, which will be the required answer. (This works because halfway towards halfway is one quarter of the way.)

Using the midpoint formula in three dimensions:

$$\begin{aligned}M &= \left(\frac{8-12}{2}, \frac{-4+0}{2}, \frac{12+8}{2}\right) \\ &= (-2, -2, 10)\end{aligned}$$

Therefore

$$\begin{aligned}N &= \left(\frac{8-2}{2}, \frac{-4-2}{2}, \frac{12+10}{2}\right) \\ &= (3, -3, 11)\end{aligned}$$

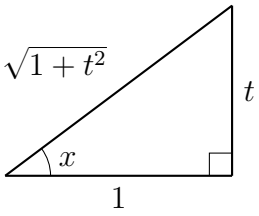
Hence (B). ✓

Question 6	**1 mark**

Solution

Let $t = \tan x$ so that $dt = (1 + t^2)dx$.

The associated right-angled triangle is shown below:



$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x + \tan^2 x}\, dx &= \int_1^{\sqrt{3}} \frac{1}{\left(\frac{t}{\sqrt{1+t^2}}\right)^2 + t^2} \times \frac{1}{1+t^2}\, dt \\
&= \int_1^{\sqrt{3}} \frac{1}{\frac{t^2}{1+t^2} + t^2} \times \frac{1}{1+t^2}\, dt \\
&= \int_1^{\sqrt{3}} \frac{1}{t^2 + t^2 + t^4}\, dt \\
&= \int_1^{\sqrt{3}} \frac{1}{2t^2 + t^4}\, dt \\
&= \int_1^{\sqrt{3}} \frac{1}{t^2(t^2+2)}\, dt
\end{aligned}$$

Hence (C). ✓

Question 7	**1 mark**

Solution

To arrive at c starting from the origin, first travel to a and then go to c.

In other words:

$$c = a + (c - a).$$

Now, $c - a$ is a scale and rotation of $b - a$ by $\frac{\pi}{2}$ radians anti-clockwise about the origin.

Recall that multiplying by i rotates a vector $\frac{\pi}{2}$ radians anti-clockwise about the origin.

Also, $\dfrac{b-a}{|b-a|}$ has unit length, so $|c-a|\dfrac{b-a}{|b-a|}$ has length $|c-a|$.

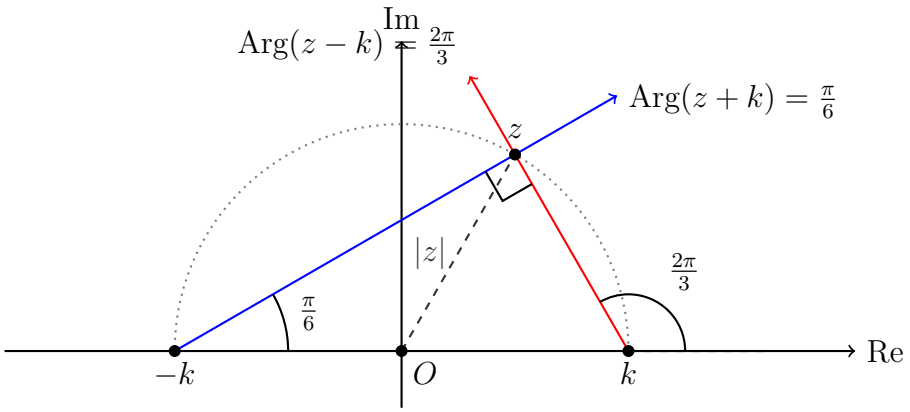
It follows that $c = a + i|c-a|\dfrac{b-a}{|b-a|}$.

Hence (A). ✓

Question 8	1 mark

Solution

The complex number z is sketched on the Argand diagram below:



Notice that the resulting triangle is right angled by angle sum of a triangle.
Hence, Pythagoras' theorem applies and we have

$$\begin{aligned}
|z-k|^2 + |z+k|^2 &= |k-(-k)|^2 \\
(z-k)\overline{(z-k)} + (z+k)\overline{(z+k)} &= 4k^2 \\
(z-k)(\overline{z}-k) + (z+k)(\overline{z}+k) &= 4k^2 \\
z\overline{z} - \cancel{k(z+\overline{z})} + k^2 + z\overline{z} + \cancel{k(z+\overline{z})} + k^2 &= 4k^2 \\
2|z|^2 &= 2k^2 \\
|z| &= k
\end{aligned}$$

We used the famous identity $|z|^2 = z\overline{z}$.

Hence (A). ✓

Note: A circle geometry solution exists using the angle in a semi-circle.
Thus $|z|$ is the radius of that semi-circle, which is k.

Question 9	**1 mark**

Solution
We will first show that options B, C and D are false.
Then we will show option A is correct.

Option B
$\exists n \in \mathbb{Z}$ such that $\forall r \in \mathbb{R}, 0 \leqslant r - n < 1$.

Counterexample to B
If such an n existed, consider $r = n+2$. This leads to the nonsensical inequality: $0 \leqslant 2 < 1$.

Option C
$\forall r \in \mathbb{R}^+$, $\exists n \in \mathbb{Z}^+$ such that $0 \leqslant r - n < 1$.

Counterexample to C
If the statement was true for all positive reals r, it would be true for $r = 0.5$.
But no positive integer n satisfies the resulting inequality: $-0.5 < n \leqslant 0.5$.

Option D
$\exists n \in \mathbb{Z}^+$ such that $\forall r \in \mathbb{R}^+, 0 \leqslant r - n < 1$.

Counterexample to D
If such an n existed, let $r = n + 2$. This leads to the nonsensical inequality: $0 \leqslant 2 < 1$.

Option A
$\forall r \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ such that $0 \leqslant r - n < 1$.

Proof of A
For any real number r, let n be the greatest integer less than or equal to r. (In the language of Question 16, $n = \lfloor r \rfloor$.) Then $r - n$ is just the fractional part of r, that is, the part of r after the decimal point. As an example, if $r = 1.723$, then $n = 1$, so $r - n = 0.723$ is the fractional part of r. This naturally lives in the interval [0,1), as required.

Hence (A). ✓

Question 10	**1 mark**

Solution

Modulus of z

$$\begin{aligned}
|z| &= \left|\left(1+\frac{i}{\sqrt{1}}\right)\left(1+\frac{i}{\sqrt{2}}\right)\cdots\left(1+\frac{i}{\sqrt{2026}}\right)\right| \\
&= \left|1+\frac{i}{\sqrt{1}}\right|\left|1+\frac{i}{\sqrt{2}}\right|\cdots\left|1+\frac{i}{\sqrt{2026}}\right| \\
&= \sqrt{1+\frac{1}{1}}\times\sqrt{1+\frac{1}{2}}\times\cdots\times\sqrt{1+\frac{1}{2026}} \\
&= \sqrt{\frac{\cancel{2}}{1}}\times\sqrt{\frac{\cancel{3}}{\cancel{2}}}\times\cdot\times\sqrt{\frac{2027}{\cancel{2026}}} \\
&= \sqrt{2027}
\end{aligned}$$

Argument of z

$$\begin{aligned}
\arg(z) &= \arg\left(\left(1+\frac{i}{\sqrt{1}}\right)\left(1+\frac{i}{\sqrt{2}}\right)\cdots\left(1+\frac{i}{\sqrt{2026}}\right)\right) \\
&= \arg\left(1+\frac{i}{\sqrt{1}}\right)+\arg\left(1+\frac{i}{\sqrt{2}}\right)+\cdots+\arg\left(1+\frac{i}{\sqrt{2026}}\right) \\
&= \arctan\left(\frac{1}{\sqrt{1}}\right)+\arctan\left(\frac{1}{\sqrt{2}}\right)+\cdots+\arctan\left(\frac{1}{\sqrt{2026}}\right) \\
&\approx \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{2026}} \\
&\approx \int_0^{2026}\frac{1}{\sqrt{x}}\,dx \\
&= [2\sqrt{x}]_0^{2026} \\
&= 2\sqrt{2026}
\end{aligned}$$

Hence (C). ✓

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Q11.

a) $(4+i)(3\overline{-2i}) = (4+i)(3+2i)$

$= 12 + 8i + 3i + 2i^2$

$= 12 + 11i - 2$

$= 10 + 11i$ (1)

Mark Allocation

(1) correct solution

Marker Feedback

- most students did this well

b) $4z^2 - 4z + 17 = 0$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(17)}}{2(4)}$$

$$= \frac{4 \pm \sqrt{-256}}{8}$$ (1)

$$= \frac{4 \pm 16i}{8}$$

$$\therefore z = \frac{1}{2} \pm 2i$$ (1)

Mark Allocation

(1) Solve for numerical z

(1) Solve for z in terms of i

Marker Feedback

- most students did this well

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Q11.

c) $|-2\sqrt{3} - 2i| = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$= 4$ (1)

$Re(-2\sqrt{3} - 2i) =$ negative

$Im(-2\sqrt{3} - 2i) =$ negative

$\therefore \tan\alpha = \left|\frac{-2}{-2\sqrt{3}}\right|$ in 3rd Quad

$\therefore$ rel $\angle = \frac{\pi}{6}$

$\therefore \alpha = \pi + \frac{\pi}{6}$

$= \frac{7\pi}{6}$

$\therefore$ Principal Argument $(-\pi < \theta \leq \pi) = -\frac{5\pi}{6}$

Positive Argument $(0 < \theta \leq 2\pi) = \frac{7\pi}{6}$ } (1)

$\therefore -2\sqrt{3} - 2i = 4e^{\left(-\frac{5\pi}{6}\right)i}$ OR $4e^{\left(\frac{7\pi}{6}\right)i}$

Mark Allocation

(1) Solves for modulus

(1) Solves for angle

Marker Feedback

- some students didn't recognise the angle had to be in quadrant 3.

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Q11

d) $\int_0^1 \frac{1-x^2}{1+x^2}\,dx = \int_0^1 \frac{2-(1+x^2)}{1+x^2}\,dx$

$= \int_0^1 \frac{2}{1+x^2} - \frac{(1+x^2)}{1+x^2}\,dx$

$= 2\int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 1\,dx$ ①

$= [2\tan^{-1}(x)]_0^1 - [x]_0^1$

$= \frac{\pi}{2} - 1$ ①

Mark Allocation

① Algebraic manipulate into an expression that can be integrated

① Final answer

Marker Feedback

•

e) $\int \frac{1}{x(\ln x)^2}\,dx$

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\therefore dx = x\,du$ ①

$= \int \frac{1}{x}\frac{1}{u^2}\,x\,du$

$= [-u^{-1}] + c$

$= \frac{-1}{\ln x} + c$ ①

Mark Allocation

① Suitable substitution

① Final answer

Marker Feedback

• some students used a substitution that didn't work

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Q11

f) $|-\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}| = \sqrt{(-1)^2+(-2)^2+(2)^2}$

$= 3$

Unit vector in this direction:

$\hat{v} = \frac{v}{|v|} = \frac{1}{3}(-\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$ ①

Vector of magnitude 12 in this direction:

$= \frac{12}{3}(-\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$= -4i - 8\underset{\sim}{j} + 8\underset{\sim}{k}$ ①

Mark Allocation

① Solves for unit vector

① Final answer

Marker Feedback

- Most students did this well

g) $|z + 3 - 2i| = |z + 1 + 2i|$

$|z - (-3+2i)| = |z - (-1-2i)|$

Equidistant from A(-3, 2) and B(-1, -2)

∴ Perpendicular bisector

AB $m = \frac{-2-2}{-1--3} = -2$

∴ ⊥ $m = \frac{1}{2}$

AB midpoint: $\left(\frac{-3-1}{2}, \frac{2+-2}{2}\right)$

$: (-2, 0)$

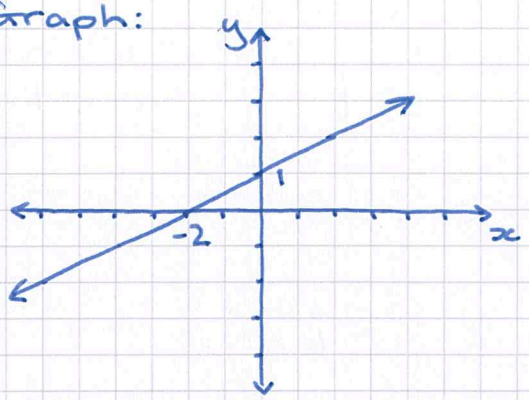
Eq. of line: $y - 0 = \frac{1}{2}(x - -2)$

$y = \frac{1}{2}x + 1$

Intercepts: y-int = 1, x-int = −2

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Graph:



ALTERNATE WORKING

Using $z = x + iy$

$|x+3+i(y-2)| = |x+1+i(y+2)|$

$(x+3)^2 + (y-2)^2 = (x+1)^2 + (y+2)^2$

$\therefore x - 2y + 2 = 0$

$\therefore y = \frac{1}{2}x + 1$

Mark Allocation

① y-intercept

① x-intercept

Marker Feedback

- Most students did this well
- some students showed no working or labelled intercepts (0/2)

h) $P(z) = z^3 - 12z^2 + 37z - 40$

One root: $2-i$, $\therefore$ Second root: $2+i$ (conjugate root theorem)

Sum of roots:

$(2-i) + (2+i) + (x+iy) = 12$ ①

$\therefore 4 + x + iy = 12$

$\therefore x + iy = 8$

$\therefore x = 8, \quad y = 0$ ①

ALTERNATE METHOD:

Form a quadratic factor from the two roots:

$(z-(2-i))(z-(2+i)) = z^2 - 4z + 5$

Divide to solve for remaining factor:

$\therefore P(z) = (z^2 - 4z + 5)(z-8)$ ①

$\therefore$ The three roots are: $(2-i)$, $(2+i)$, (8) ①

Mark Allocation

① Valid method ① Final answer

Marker Feedback

- Most students did this well

Question 12 (15 marks) Start a new page.

(a) Show that ${}^nP_r$ is divisible by r.

$${}^nP_r = \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \underbrace{n(n-1)\cdots(n-r+1)}_{n \to (n-r+1)}$$

i.e $n-(n-r+1)+1$ consec. integers multiplied $= (r)$

$\therefore$ There are r consec. integers, hence ${}^nP_r$ is div. by r.

Mark	Criteria
2	Correct solution.
1	Shows that $nP_r = n(n-1)(n-2)\ldots(n-r+1)$ OR Makes an argument that the expression is due to combinatorics without connecting the formula to nC_r.

Comments:

Many students were able to obtain 1 mark but failed to obtain full marks due to a poor or unclear final argument. Students needed to clearly state that since there are r consecutive integers, the product must be divisible by r .

Alt. Sol^n

$${}^nP_r = \frac{n!}{(n-r)!} = \binom{n}{r} r!$$

since $\binom{n}{r}, n! \in \mathbb{Z}$

$$\therefore {}^nP_r = \binom{n}{r}(r)(r-1)!$$

which must be div. by r as r is a factor.

(b) Use the identities $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ and $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ to prove that $1 + \tan^2\theta = \sec^2\theta$.

$$LHS = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{\frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2 - \frac{1}{4}\left(e^{i\theta}-e^{-i\theta}\right)^2 \times 4}{\frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2 \times 4}$$

$$= \frac{e^{2i\theta} + 2 + e^{-2i\theta} - \left(e^{2i\theta} - 2 + e^{-2i\theta}\right)}{e^{2i\theta} + 2 + e^{-2i\theta}}$$

$$= \frac{4}{\left(e^{i\theta}+e^{-i\theta}\right)^2} = \frac{4}{(2\cos\theta)^2} = \frac{4}{4\cos^2\theta} = \sec^2\theta = RHS$$

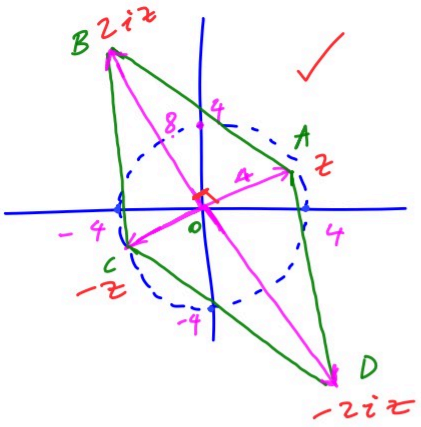
as required.

Mark	Criteria
2	Correct solution.
1	Substitutes in the given identities $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ and $\cos\theta = \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)$

Comments:

Done well. To obtain full marks, students had to clearly show how they obtained LHS or RHS.

(c) Find the area of the quadrilateral formed by $z, -z, 2iz$ and $-2iz$ if $|z| = 4$.



OACD is a kite.

$A = \frac{1}{2} d_1 d_2$

$= \frac{1}{2}(2|z|)(2|2iz|)$

$= \frac{1}{2}(8)(8\times 2)$

$= 64 \text{ u}^2$

Mark	Criteria
2	Correct solution.
1	Makes significant progress with aide of a diagram **and** uses the given modulus $\|z\| = 4$.

Comments:

Many students did not arrive at the correct solution due to incorrectly drawing the given terms. The most successful students correctly represented these expressions as vectors from the origin and clearly used the given modulus to establish that the quadrilateral formed is a kite.

(d) Evaluate $\int_{-6}^{3} \frac{1}{\sqrt{45 - 12x - x^2}}\, dx$.

$= \int_{-6}^{3} \frac{1}{\sqrt{81 - (x+6)^2}}\, dx$

$36+45$

$= \left[\sin^{-1}\left(\frac{x+6}{9}\right)\right]_{-6}^{3}$

$= \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2}$

Mark	Criteria
2	Correct solution.
1	Obtains the correct antiderivative. OR Correctly substitutes in the limits.

Comments:

A number of students made a range of errors when completing the square and obtaining the correct antiderivative. These included the following:

- Obtaining $\tan^{-1}$ instead of $\sin^{-1}$
- Obtaining $\sin^{-1}\left(\frac{x+6}{3}\right)$
- Expressing the numerator in the form $\sqrt{c^2 + (x-a)^2}$ $c, a \in \mathbb{R}$
- Arithmetic errors with completing the square.

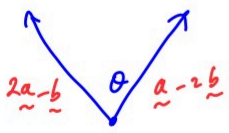
Students are reminded to double-check that their expression is correct when completing the square.

(e) Let $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

It is given that $|2\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{a} - 2\underset{\sim}{b}| = \sqrt{10}$.

Calculate the exact angle between $2\underset{\sim}{a} - \underset{\sim}{b}$ and $\underset{\sim}{a} - 2\underset{\sim}{b}$.

let the angle be θ



$\cos\theta = \frac{(2\underset{\sim}{a} - \underset{\sim}{b}).(\underset{\sim}{a} - 2\underset{\sim}{b})}{|2\underset{\sim}{a} - \underset{\sim}{b}||\underset{\sim}{a} - 2\underset{\sim}{b}|}$

$= \frac{\begin{pmatrix} 4-1 \\ -2+1 \\ 2-2 \end{pmatrix} \cdot \begin{pmatrix} 2-2 \\ -1+2 \\ 1-4 \end{pmatrix}}{\sqrt{10}\sqrt{10}}$

$= \frac{\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}}{10} = \frac{-1}{10}$

$\theta = \cos^{-1}\left(\frac{-1}{10}\right)$ or $-\cos^{-1}\left(\frac{1}{10}\right)$

OR $\pi - \cos^{-1}\left(\frac{1}{10}\right)$

Mark	Criteria
2	Correct solution.
1	Makes some correct progress in simplifying their expression using vector algebra. OR Makes an error in obtaining their cos expression.

Comments:

Students who avoided using vector algebra such as $a \cdot a = |a|^2$ were generally more successful in obtaining full marks. Many errors resulted from arithmetic mistakes and a lack of care when dealing with negative signs.

(f) A particle of mass 6 kg moves in a straight line subject to a force $F = \frac{1}{v+3}$ newtons. At time t seconds, the displacement of the particle is x metres and the velocity is v ms^{-1}.
The particle was initially at rest at the origin.
Find x when $v = 4$ ms^{-1}.

$F = 6\ddot{x} = \frac{1}{v+3} \Rightarrow \ddot{x} = \frac{1}{6(v+3)}$ ✓

Relating v and x

$v\frac{dv}{dx} = \frac{1}{6(v+3)}$

initially
$t=0$
$x=0$
$v=0$
$x=?\ V=4$

$6\int_0^4 v^2+3v\,dv = \int_0^x dx$

$[x]_0^x = 6\left[\frac{1}{3}v^3 + \frac{3}{2}v^2\right]_0^4$

$x = 6\left(\frac{1}{3}4^3 + \frac{3}{2}4^2\right)$

$= 272$ ✓

Mark	Criteria
2	Correct solution.
1	Obtains $a = \frac{1}{6(x+3)}$ Or Makes substantial correct progress.

Comments:

Done well. Some errors occurred as a result of incorrectly substituting and obtaining the wrong antiderivative.

(g) Find $\int \frac{8x+2}{(3x-2)(x^2+2)}\,dx$.

$\frac{8x+2}{(3x-2)(x^2+2)} \equiv \frac{A}{3x-2} + \frac{Bx+C}{x^2+2}$

$8x+2 \equiv A(x^2+2) + (Bx+C)(3x-2)$

$x=\frac{2}{3} \Rightarrow \frac{22}{3} = A\left(\frac{22}{9}\right) \Rightarrow \boxed{A=3}$

$x=0 \Rightarrow 2 = 3(2) + C(-2)$

$\boxed{C=2}$

$x=1 \Rightarrow 10 = 3(3) + (B+2)(1)$

$B+2=1 \Rightarrow \boxed{B=-1}$ ✓

$\therefore I = \int \frac{3}{3x-2} + \frac{-x+2}{x^2+2}\,dx$

$= \ln|3x-2| - \frac{1}{2}\int\frac{2x}{x^2+2}dx + 2\int\frac{1}{x^2+2}dx$ ✓

$= \ln|3x-2| - \frac{1}{2}\ln|x^2+2| + \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$ ✓

Mark	Criteria
3	Correct solution.
2	Applies the correct algebraic manipulations after successfully performing partial fraction decomposition.
1	Correctly obtains the values of a, b and c or alternatively using partial fractions.

Comments:

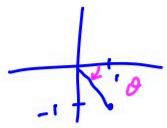
Done well. The main area in which students lost marks was the partial fraction decomposition. Students are reminded to apply the most appropriate method depending on the expression. In this case, substitution was far more efficient than equating coefficients, which required additional algebra.

Question 13 (15 marks) Start a new page.

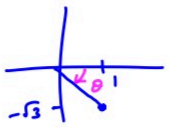
(a) By writing $1-i$ and $1-i\sqrt{3}$ in mod-arg form, find the exact value of $\sin\frac{7\pi}{12}$.

$1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ ①

$1-i\sqrt{3} = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ ②



$\theta = \tan^{-1}(-1) = -\pi/4$



$\theta = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$ ✓

① × ②: $(1-i)(1-i\sqrt{3}) = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}-\frac{\pi}{3}\right)$

$1-\sqrt{3}-i-i\sqrt{3} = 2\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{12}\right)$ ✓

Equate im. parts

$2\sqrt{2}\sin\left(-\frac{7\pi}{12}\right) = -1-\sqrt{3}$

$-2\sqrt{2}\sin\frac{7\pi}{12} = -1-\sqrt{3}$ since $\sin x$ is odd

$\therefore \sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ OR $\frac{\sqrt{2}+\sqrt{6}}{4}$ ✓

Mark	Criteria
3	Correct solution.
2	Divides or multiplies both complex numbers and then equates the imaginary part.
1	Correctly expresses both complex numbers in mod-arg form. OR Shows some correct progress.

Comments:
Done well. To avoid errors when determining the argument, students are reminded to draw a diagram. Additionally, students should take care when performing arithmetic involving complex numbers.

(b) It is given that if x^2 divides y^2 then x divides y for all integers x and y.
Suppose that n is a positive integer and that $\sqrt{n}$ is not an integer.
Prove by contradiction that $\sqrt{n}$ is irrational.

Assume that $\sqrt{n}$ is rational

i.e $\sqrt{n} = \frac{p}{q}$ p, q are coprime ✓

$n = \frac{p^2}{q^2}$

But $n \in \mathbb{Z}^+$ (given)

$\therefore q^2$ must divide p^2 } given in 1st sentence ✓

$\Rightarrow q$ divides p.

Hence $\sqrt{n} = \frac{p}{q} = k$ where $k \in \mathbb{Z}$

But this contradicts that $\sqrt{n}$ is NOT an integer.

$\therefore \sqrt{n}$ cannot be expressed in the form p/q and must be irrational. ✓

Mark	Criteria
3	Correct solution.
2	Makes an error in their proof OR Fails to clearly justify their reasoning.
1	Assumes that $\sqrt{n}$ is a rational number in the form $\frac{p}{q}$ where p and q are coprime.

Comments:
Many students were penalised for incorrectly assuming that if a square number p^2 is divisible by a number n, then p must also be divisible by n . The problem with this assumption is that it only holds true if n is a prime number. Many students were still able to obtain 2 out of the 3 marks despite making this assumption as this was treated as an error carried forward, provided that their subsequent reasoning and justification were clear and logically valid.

(c) Evaluate $\int_1^e 1 \cdot \cos(\ln x)\,dx$.

int. diff.

$v' = 1 \quad v = x$

$u = \cos(\ln x) \quad u' = -\frac{1}{x}\sin(\ln x)$

$$I = x\cos(\ln x)\Big|_1^e + \int_1^e x \cdot \frac{1}{x}\sin(\ln x)\,dx$$

int. diff.

$v' = 1 \quad v = x$

$u = \sin(\ln x) \quad u' = \frac{1}{x}\cos(\ln x)$

$$= e\cos(1) - 1\cos(0) + \left(x\sin(\ln x)\Big|_1^e - \int_1^e \frac{x}{x}\cos(\ln x)\,dx\right)$$

$$= e\cos 1 - 1 + (e\sin 1 - 0 - I)$$

$$I = e\cos 1 - 1 + e\sin 1 - I$$

$$2I = e\cos 1 - 1 + e\sin 1$$

$$I = \frac{e\cos 1 - 1 + e\sin 1}{2}$$

Mark	Criteria
3	Correct solution.
2	Makes an error in their integration by parts.
1	Makes some correct progress.

Comments:

Many students were penalised for failing to substitute 0 into their expression. Additionally, a number of errors arose as a result of students not knowing which part to integrate and which to differentiate when it came to setting up their integration by parts.

(d) The points A, B and P have position vectors $\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + \lambda\underset{\sim}{k}$, $\overrightarrow{OB} = 5\underset{\sim}{i} + \lambda\underset{\sim}{j}$ and $\overrightarrow{OP} = -\underset{\sim}{i} - \lambda\underset{\sim}{j} + \underset{\sim}{k}$ respectively.

Find the possible values of λ if $\angle AOP = \angle BOP$.



$$\cos\alpha = \frac{\overrightarrow{OA}\cdot\overrightarrow{OP}}{|\overrightarrow{OA}||\overrightarrow{OP}|} = \frac{\begin{pmatrix}3\\4\\\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\-\lambda\\1\end{pmatrix}}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}} = \frac{-3-4\lambda+\lambda}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}}$$

$$\cos\beta = \frac{\overrightarrow{OB}\cdot\overrightarrow{OP}}{|\overrightarrow{OB}||\overrightarrow{OP}|} = \frac{\begin{pmatrix}5\\\lambda\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-\lambda\\1\end{pmatrix}}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}} = \frac{-5-\lambda^2}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}}$$

Equate $\cos\alpha$, $\cos\beta$

$$\Rightarrow \frac{-3-3\lambda}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}} = \frac{-5-\lambda^2}{\sqrt{25+\lambda^2}\sqrt{2+\lambda^2}}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1) = 0 \Rightarrow \lambda = 1, 2$$

Mark	Criteria
3	Correct solution.
2	Makes an error in their working without making the question easier.
1	Makes some correct progress.

Comments:

Many students made errors in their vector algebra. Students are reminded to be more organised when approaching these questions and to take extreme care when dealing with negative signs.

(e) A particle is moving in simple harmonic motion with a period of 6π seconds according to the equation $x = a\cos(nt + \alpha)$, where x is the displacement in metres, $a > 0$, and $0 \leqslant \alpha < 2\pi$.

When $t = 2$, the particle's displacement is $x = -2$ metres and its velocity is $v = 1$ metre per second.

Find a and α correct to four significant figures.

$T = 6\pi = \frac{2\pi}{n} \Rightarrow n = \frac{1}{3}$

$x = a\cos\left(\frac{t}{3} + \alpha\right)$

$v = -\frac{a}{3}\sin\left(\frac{t}{3} + \alpha\right)$

$t = 2,\ x = -2 \Rightarrow -2 = a\cos\left(\frac{2}{3} + \alpha\right)$ ①

$t = 2,\ v = 1 \Rightarrow 1 = -\frac{a}{3}\sin\left(\frac{2}{3} + \alpha\right)$

$3 = -a\sin\left(\frac{2}{3} + \alpha\right)$ ②

① ^{2} + ② ^{2} $\Rightarrow 4 + 9 = a^2\left[\cos^2\left(\frac{2}{3} + \alpha\right) + \sin^2\left(\frac{2}{3} + \alpha\right)\right]$

$\therefore a = \sqrt{13} \doteqdot 3.606$ since $a > 0$

② ÷ ① $\Rightarrow -\tan\left(\frac{2}{3} + \alpha\right) = \frac{3}{-2}$

$\tan\left(\frac{2}{3} + \alpha\right) = \frac{3}{2}$

Ref. $\angle = \text{Tan}^{-1}\left(\frac{3}{2}\right)$

Adjust Domain

$\frac{2}{3} < \frac{2}{3} + \alpha \leq 2\pi + \frac{2}{3}$ ($\doteqdot 6.95$)

$\frac{2}{3} + \alpha = \text{Tan}^{-1}\left(\frac{3}{2}\right)$, $\pi + \text{Tan}^{-1}\left(\frac{3}{2}\right)$, $2\pi + \text{Tan}^{-1}\left(\frac{3}{2}\right)$, ...

$\doteqdot 0.98$; $\doteqdot 4.12$; $\doteqdot 7.26$

Reject as $a\cos\left(\frac{2}{3} + \alpha\right)$ must be positive.

Reject as $a\cos\left(\frac{2}{3} + \alpha\right)$ must be positive.

$\therefore \alpha = \pi + \text{Tan}^{-1}\left(\frac{3}{2}\right) - \frac{2}{3} \doteqdot 3.468$

Mark	Criteria
3	Correct solution.
2	Correctly finds a OR Correctly finds α with a clear justification based on the boundary conditions. OR Obtains the incorrect value for α but carries forward their error to find a.
1	Substitutes in the initial conditions ($t = 2$) into both x and v.

Comments:

To obtain full marks, students needed to justify why α must lie in the 3rd quadrant. Many students obtained only 1 or 2 marks, with only a handful obtaining all 3 marks. A common error was assuming that α is in the 1st quadrant which cannot be true when considering the boundary conditions when $t = 2$.

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Q14.

a) $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} \leq \frac{3}{2} - \frac{1}{2n^2}$ $\quad n > 0$

① Verify base case:

$$\frac{1}{(1)^3} \leq \frac{3}{2} - \frac{1}{2(1)^3}$$

$$1 \leq 1$$

$\therefore$ Identity holds true for base case. ①

② Assume true for $n = k$

$$\therefore \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} \leq \frac{3}{2} - \frac{1}{2k^2}, \quad k > 0 \quad *$$

③ Prove true for $n = k+1$

R.T.P $\underbrace{\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3}}_{A} + \frac{1}{(k+1)^3} \leq \underbrace{\frac{3}{2} - \frac{1}{2(k+1)^2}}_{B}$

from inductive assumption *

$$\underbrace{\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3}}_{A} \left(+ \frac{1}{(k+1)^3} \right) \leq \underbrace{\frac{3}{2} - \frac{1}{2k^2}}_{C} \left(+ \frac{1}{(k+1)^3} \right), \quad k > 0$$

$\therefore$ R.T.P $\frac{3}{2} - \frac{1}{2k^2} + \frac{1}{(k+1)^3} \leq \frac{3}{2} - \frac{1}{2(k+1)^2}$ ① $(A \leq C \leq B)$

$$\therefore \frac{1}{(k+1)^3} \leq \frac{1}{2k^2} - \frac{1}{2(k+1)^2}$$

$$\therefore \frac{1}{(k+1)^3} \leq \frac{2k+1}{2k^2(k+1)^2}, \quad k > 0$$

$$\therefore 2k^2(k+1)^2 \leq (2k+1)(k+1)^3, \quad (k+1)^2 > 0$$

$$\therefore 2k^2 \leq (2k+1)(k+1)$$

$$\therefore 2k^2 \leq 2k^2 + 3k + 1$$

Known: $3k + 1 \geq 0$, $k > 0$

$\therefore$ By Principle of Mathematical Induction the identity is true for $n > 0$. ①

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ALTERNATE METHOD (from part 3)

③ Prove true for $n = k+1$

R.T.P. $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{(k+1)^3} \leq \frac{3}{2} - \frac{1}{2(k+1)^2}$

$(LHS - RHS \leq 0)$

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{(k+1)^3} - \left(\frac{3}{2} - \frac{1}{2(k+1)^2}\right) \leq 0$$

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{(k+1)^3} - \frac{3}{2} + \frac{1}{2(k+1)^2} \leq 0$$

from inductive assumption *

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{(k+1)^3} - \frac{3}{2} + \frac{1}{2(k+1)^2} \leq \frac{3}{2} - \frac{1}{2k^2} + \frac{1}{(k+1)^3} - \frac{3}{2} + \frac{1}{2(k+1)^2}$$

$$\leq -\frac{1}{2k^2} + \frac{1}{(k+1)^3} + \frac{1}{2(k+1)^2}$$

$$= \frac{-(k+1)^3 + 2k^2 + k^2(k+1)}{2k^2(k+1)^3}$$

$$= \frac{-(k^3+3k^2+3k+1) + 2k^2 + k^3 + k^2}{2k^2(k+1)^3}$$

$$= \frac{-(3k+1)}{2k^2(k+1)^3}, \quad k > 0$$

$$\leq 0$$

∴ By Principle of Mathematic Induction the identity is true for $n > 0$. (1)

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Mark Allocation

① Verify base case

① Valid use of inductive hypothesis

① Valid and reasoned algebraic working to reach final proof statement

Marker Feedback

- many students use of inductive hypothesis was messy and poorly set out
- some students algebraic proof in step 3 was very messy, poorly set out, and hard to follow
- some student subbed in the assumption as an equivalence ($=$) when it should initially be ($\leq$)
- some students made clear algebraic errors in step ③ which I deducted a mark for

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Q14.

b) $\int_0^{\pi/4} \sin^3 x \sec^6 x \, dx$

$= \int_0^{\pi/4} \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^3 x} \, dx$

$= \int_0^{\pi/4} \tan^3 x \cdot \sec^3 x \, dx$

$= \int_0^{\pi/4} \tan^2 x \cdot \sec^2 x \cdot \tan x \sec x \, dx$ (1)

$= \int_0^{\pi/4} (\sec^2 x - 1) \sec^2 x \cdot \tan x \sec x \, dx$

$= \int_0^{\pi/4} \sec^4 x \cdot \tan x \sec x - \sec^2 x \tan x \sec x \, dx$

$= \left[\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} \right]_0^{\pi/4}$ (1)

$= \left(\frac{1}{5\left(\cos\frac{\pi}{4}\right)^5} - \frac{1}{3\left(\cos\frac{\pi}{4}\right)^3} \right) - \left(\frac{1}{5(\cos 0)^5} - \frac{1}{3(\cos 0)^3} \right)$

$= \left(\frac{1}{5\left(\frac{1}{4\sqrt{2}}\right)} - \frac{1}{3\left(\frac{1}{2\sqrt{2}}\right)} \right) - \left(\frac{1}{5} - \frac{1}{3} \right)$

$= \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} \right) - \left(\frac{3-5}{15} \right)$

$= \frac{12\sqrt{2} - 10\sqrt{2}}{15} + \frac{2}{15}$

$= \frac{2\sqrt{2} + 2}{15}$

$= \frac{2(\sqrt{2}+1)}{15}$ (1)

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ALTERNATE METHOD (u-sub options)

$$\int_0^{\pi/4} (\sec^2 x - 1)\sec^2 x \cdot \tan x \sec x \, dx$$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

Bounds:

when $x = \frac{\pi}{4}$, $u = \sqrt{2}$

$x = 0$, $u = 1$

$$= \int_1^{\sqrt{2}} (u^2 - 1)u^2 \, du$$

$$= \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\sqrt{2}} \quad (1)$$

$$= \frac{2(\sqrt{2}+1)}{15} \quad (1)$$

OR

Let $u = \cos x$, $\sin^2 x = 1 - u^2$

$dx = \frac{du}{-\sin x}$

Bounds:

when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$

$x = 0$, $u = 1$

$$\int_0^{\pi/4} \sin^3 x \sec^6 x \, dx = \int_1^{\frac{1}{\sqrt{2}}} -(1-u^2)\frac{1}{u^6} \, du$$

$$= \int_{\frac{1}{\sqrt{2}}}^1 u^{-6} - u^{-4} \, du$$

$$= \left[-\frac{u^{-5}}{5} + \frac{u^{-3}}{3} \right]_{\frac{1}{\sqrt{2}}}^1 \quad (1)$$

$$= \left(-\frac{1}{5} + \frac{1}{3} \right) - \left(-\frac{1}{5\left(\frac{1}{4\sqrt{2}}\right)} + \frac{1}{3\left(\frac{1}{2\sqrt{2}}\right)} \right)$$

$$= \frac{2}{15} - \left(\frac{-4\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} \right)$$

$$= \frac{2}{15} - \left(\frac{-2\sqrt{2}}{15} \right)$$

$$= \frac{2\sqrt{2}+2}{15}$$

$$= \frac{2(\sqrt{2}+1)}{15} \quad (1)$$

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Marking Allocation

(1) Initial manipulation of trig identities to create an expression that can be integrated

(1) Integration

(1) Final solution

Marker Feedback

- many students did this well, although a few algebraic mistakes were made
- many students didn't simplify their final answer. I didn't penalise, but it is good practice to simplify

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Q14.

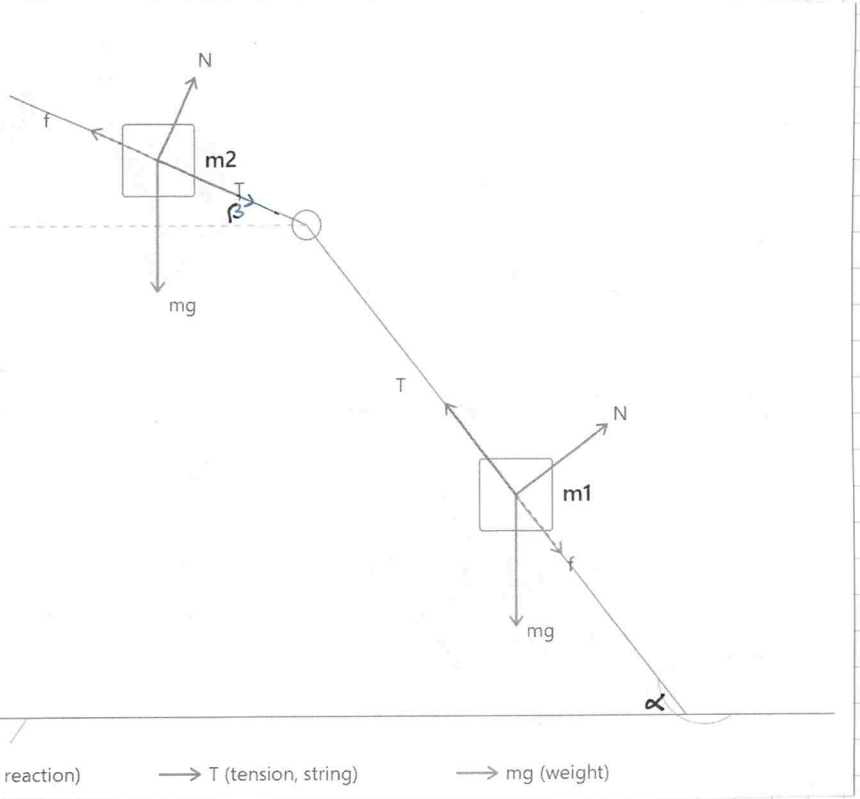
c)

Weight force components parallel to slope:

m_2:
$m_2 g \sin\beta$

m_1:
$m_1 \sin\alpha$

} down slope



Note: acceleration force is in the direction of down the slope (for both boxes)

m_2 mass:

$m_2 g \sin\beta + T = m_2 a$ ——— ① (1)

m_1 mass:

$m_1 g \sin\alpha - T = m_1 a$ ——— ② (1)

① + ②

$m_2 g \sin\beta + m_1 g \sin\alpha = (m_2 + m_1) a$

$\therefore a = \frac{g}{m_2 + m_1} (m_2 \sin\beta + m_1 \sin\alpha)$ (1)

Mark Allocation

(1) Correct force balanced equation for m_2
(1) Correct force balanced equation for m_1
} or very clearly labelled on force diagram

(1) Correct simultaneous equation proof

Note: for full marks Tension had to be included

Marker Feedback

- Tension direction confused many students

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Q14.

d) i) RTP: $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Note: $(\cos x + i\sin x)^7 = \cos(7x) + i\sin(7x)$

$$\text{LHS} = \binom{7}{0}c^7 + \binom{7}{1}c^6\underbrace{(is)^1}_{i} + \binom{7}{2}c^5\underbrace{(is)^2}_{i^2=-1} + \binom{7}{3}c^4\underbrace{(is)^3}_{i^3=-i} + \binom{7}{4}c^3\underbrace{(is)^4}_{i^4=1} + \binom{7}{5}c^2\underbrace{(is)^5}_{i^5=i}$$
$$+ \binom{7}{6}c^1\underbrace{(is)^6}_{i^6=-1} + \binom{7}{7}\underbrace{(is)^7}_{i^7=-i} \quad (1)$$

Equate real parts:

$\cos(7x) = 1\cos^7x - 21\cos^5x\sin^2x + 35\cos^3x\sin^4x - 7\cos x\sin^6x$ (1)

$\sin^2x = 1 - \cos^2x$

$\sin^4x = (1-\cos^2x)^2 = 1 - 2\cos^2x + \cos^4x$

$\sin^6x = (1-\cos^2x)^3 = 1 - 3\cos^2x + 3\cos^4x - \cos^6x$

$\therefore$ RHS $= \cos^7x - 21\cos^5x(1-\cos^2x) + 35\cos^3x(1 - 2\cos^2x + \cos^4x)$
$- 7\cos x(1 - 3\cos^2x + 3\cos^4x - \cos^6x)$

$= \cos^7x - 21\cos^5x + 21\cos^7x + 35\cos^3x - 70\cos^5x + 35\cos^7x$
$- 7\cos x + 21\cos^3x - 21\cos^5x + 7\cos^7x$

$= 64\cos^7x - 112\cos^5x + 56\cos^3x - 7\cos x$

$\therefore \cos(7\theta) = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$ (1)

Mark Allocation

(1) Expansion

(1) Equating real parts

(1) Correct algebraic working for final step of proof

Marker Feedback

- Most students did this well

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Q14.

d) ii) $x^3 - 7x^2 + 14x - 7 = 0$ ——— ①

From i)

$$\cos(7\theta) = \cos\theta\left(64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7\right)$$

$$\therefore \frac{\cos(7\theta)}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

let $x = 4\cos^2\theta$

From ①

$$(4\cos^2\theta)^3 - 7(4\cos^2\theta)^2 + 14(\cos^2\theta) - 7 = 0$$

$$\therefore 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7 = 0 \quad ①$$

$$\therefore \frac{\cos(7\theta)}{\cos\theta} = 0 \quad , \quad \cos\theta \neq 0$$

$$\therefore \cos(7\theta) = \frac{\pi}{2} + k\pi \text{ , for } k \in \mathbb{Z}$$

$$\therefore 7\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \theta = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}$$

$$\therefore x = 4\cos^2\left(\frac{\pi}{14}\right), 4\cos^2\left(\frac{3\pi}{14}\right), 4\cos^2\left(\frac{5\pi}{14}\right) \quad ①$$

Mark Allocation:

① Suitable substitution has been substituted

① Algebraically correct final answer

Marker Feedback

- some students didn't solve for 3 distinct roots

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Q14.

d) iii) sum of roots for: $x^3 - 7x^2 + 14x - 7 = 0$

$$\alpha + \beta + \gamma = \frac{-(-7)}{1} = 7$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{14}{1} = 14$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$$

$$= 7^2 - 2(14)$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = 21$$

Let $\alpha = 4\cos^2\left(\frac{\pi}{14}\right)$, $\beta = 4\cos^2\left(\frac{3\pi}{14}\right)$, $\gamma = 4\cos^2\left(\frac{15\pi}{14}\right)$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = \left(4\cos^2\left(\frac{\pi}{14}\right)\right)^2 + \left(4\cos^2\left(\frac{3\pi}{14}\right)\right)^2 + \left(4\cos^2\left(\frac{15\pi}{14}\right)\right)^2$$

$$= 16\cos^4\left(\frac{\pi}{14}\right) + 16\cos^4\left(\frac{3\pi}{14}\right) + 16\cos^4\left(\frac{15\pi}{14}\right)$$

$$= 16\left(\cos^4\left(\frac{\pi}{14}\right) + \cos^4\left(\frac{3\pi}{14}\right) + \cos^4\left(\frac{15\pi}{14}\right)\right)$$

$$= 21$$

$$\therefore \cos^4\left(\frac{\pi}{14}\right) + \cos^4\left(\frac{3\pi}{14}\right) + \cos^4\left(\frac{15\pi}{14}\right) = \frac{21}{16}$$

Mark Allocation

① Correct answer

Marker Feedback

- some students neglected the $(\ )^4$

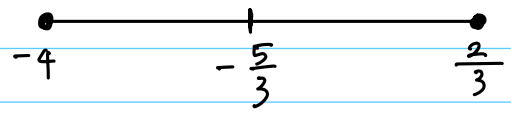
Question 15

(a) Let $v = 0$

$$3x^2 + 10x - 8 = 0$$

$$(3x-2)(x+4) = 0$$

$$x = \frac{2}{3}, -4$$



$\therefore$ C.O.M $x = -\frac{5}{3}$ ✓

$a = \frac{7}{3}$ ✓

Marker's Comments

- Generally well done
- Some students also obtained the centre of motion and amplitude using the alternative method below:

$$v^2 = -3\left(x^2 + \frac{10}{3}x - \frac{8}{3}\right)$$

$$= -3\left(\left(x + \frac{5}{3}\right)^2 - \frac{49}{9}\right)$$

$$= 3\left(\frac{49}{9} - \left(x + \frac{5}{3}\right)^2\right)$$

$$v^2 = n^2\left(a^2 - (x-c)^2\right)$$

$$\therefore a = \frac{7}{3}, \quad c = -\frac{5}{3}$$

(b)

(i) $\text{Sum} = \dfrac{(-\omega)^n - 1}{-\omega - 1}$ ✓

$= \dfrac{-\omega^n - 1}{-\omega - 1}$ (as n is odd)

$= \dfrac{\omega^n + 1}{\omega + 1}$

$= \dfrac{\text{cis}\, 2\pi + 1}{\text{cis}\, \frac{2\pi}{n} + 1}$

$= \dfrac{2}{\text{cis}\, \frac{2\pi}{n} + 1}$

$= \dfrac{2\,\text{cis}\left(-\frac{\pi}{n}\right)}{\text{cis}\, \frac{\pi}{n} + \text{cis}\left(-\frac{\pi}{n}\right)}$

$= \dfrac{2\left(\cos\frac{\pi}{n} - i\sin\frac{\pi}{n}\right)}{2\cos\frac{\pi}{n}}$

as [cos is even, sin is odd]

[$z + \bar{z} = 2\,\text{Re}(z)$]

$= 1 - i\tan\frac{\pi}{n}$ ✓

Marker's Comments

- Not well done. The easiest way to prove the identity was to recognise that the sum was the sum of a geometric progression.

(ii) $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \dfrac{\omega^n - 1}{\omega - 1}$

$$= \frac{\text{cis}\,2\pi - 1}{\omega - 1}$$

$$= 0 \qquad (1)$$

$1 - \omega + \omega^2 - \ldots + \omega^{n-1} = 1 - i\tan\frac{\pi}{n}$ (2)

(1) + (2) :

$2\left(1 + \omega^2 + \omega^4 + \ldots + \omega^{n-1}\right) = 1 - i\tan\frac{\pi}{n}$

$\therefore\ 1 + \omega^2 + \omega^4 + \cdots + \omega^{n-1} = \frac{1}{2}\left(1 - i\tan\frac{\pi}{n}\right)$ ✓

Alternatively to prove (1),

$\omega^n - 1 = 0$

$(\omega - 1)\left(1 + \omega + \omega^2 + \cdots + \omega^{n-1}\right) = 0$

As $\omega \neq 1$,

$\therefore\ 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

Marker's Comments

- Not well done. Students should be familiar with the technique of adding two sums. This technique is one of the techniques needed to prove certain identities in Y11 Ext 1 Binomial Theorem.

(c) $I_n = \int_0^r x^{n+\frac{1}{2}}(r-x)^{\frac{1}{2}}\,dx$ $\quad u = x^{n+\frac{1}{2}}$ $\quad v' = (r-x)^{\frac{1}{2}}$

$u' = (n+\frac{1}{2})x^{n-\frac{1}{2}}$ $\quad v = \frac{-2(r-x)^{\frac{3}{2}}}{3}$

$$= \left[-\frac{2}{3}x^{n+\frac{1}{2}}(r-x)^{\frac{3}{2}}\right]_0^r + \frac{2}{3}(n+\frac{1}{2})\int_0^r x^{n-\frac{1}{2}}(r-x)^{\frac{3}{2}}\,dx \quad \checkmark$$

$$= \frac{2n+1}{3}\int_0^r x^{n-\frac{1}{2}}(r-x)^{\frac{1}{2}}(r-x)\,dx \quad \checkmark$$

expand

$$= \frac{2n+1}{3}\int_0^r r x^{n-\frac{1}{2}}(r-x)^{\frac{1}{2}} - x^{n+\frac{1}{2}}(r-x)^{\frac{1}{2}}\,dx$$

$$I_n = \frac{2n+1}{3}\left[r\,I_{n-1} - I_n\right]$$

$$\therefore \frac{2n+4}{3}I_n = \frac{(2n+1)r}{3}I_{n-1}$$

$$\therefore I_n = \frac{(2n+1)r}{2n+4}I_{n-1} \quad \checkmark$$

Marker's Comments

- Generally well done

(d)

(i) Solve simultaneously:

$$\left| \underset{\sim}{a} + \lambda \underset{\sim}{b} - \underset{\sim}{c} \right| = R$$

$$\therefore \left| \underset{\sim}{a} + \lambda \underset{\sim}{b} - \underset{\sim}{c} \right|^2 = R^2 \quad \checkmark$$

$$\left((\underset{\sim}{a} - \underset{\sim}{c}) + \lambda \underset{\sim}{b} \right) \cdot \left((\underset{\sim}{a} - \underset{\sim}{c}) + \lambda \underset{\sim}{b} \right) = R^2$$

$$\therefore |\underset{\sim}{a} - \underset{\sim}{c}|^2 + 2\lambda \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) + \lambda^2 |\underset{\sim}{b}|^2 = R^2$$

$$\therefore |\underset{\sim}{b}|^2 \lambda^2 + 2\lambda \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) + |\underset{\sim}{a} - \underset{\sim}{c}|^2 - R^2 = 0 \quad \checkmark$$

(ii) $\Delta = 0$

$$\therefore 4\left(\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) \right)^2 - 4 |\underset{\sim}{b}|^2 \left(|\underset{\sim}{a} - \underset{\sim}{c}|^2 - R^2 \right) = 0 \quad \checkmark$$

$$\left(\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) \right)^2 = |\underset{\sim}{b}|^2 \left(|\underset{\sim}{a} - \underset{\sim}{c}|^2 - R^2 \right)$$

$$\left(\frac{\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})}{|\underset{\sim}{b}|} \right)^2 = |\underset{\sim}{a} - \underset{\sim}{c}|^2 - R^2$$

$$\therefore R^2 = |\underset{\sim}{a} - \underset{\sim}{c}|^2 - \left(\frac{\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})}{|\underset{\sim}{b}|} \right)^2 \quad \checkmark$$

(iii) $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = |\underset{\sim}{b}||\underset{\sim}{a} - \underset{\sim}{c}| \cos\theta$

$$\therefore R^2 = |\underset{\sim}{a} - \underset{\sim}{c}|^2 - \left(\frac{|\underset{\sim}{b}||\underset{\sim}{a} - \underset{\sim}{c}| \cos\theta}{|\underset{\sim}{b}|} \right)^2$$

$$= |\underset{\sim}{a} - \underset{\sim}{c}|^2 \left(1 - \cos^2\theta\right)$$

$$= |\underset{\sim}{a} - \underset{\sim}{c}|^2 \sin^2\theta$$

$$\therefore R = |\underset{\sim}{a} - \underset{\sim}{c}| \sin\theta$$ ✓

(iv) Consider a sphere centred at a point on $\underset{\sim}{r}_2$

ie. $\underset{\sim}{c} = \begin{pmatrix} 2 + 5\mu \\ 5 \\ -3 - 3\mu \end{pmatrix}$ ✓

From $\underset{\sim}{r}_1$, $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \underset{\sim}{a} - \underset{\sim}{c} = \begin{pmatrix} -5\mu \\ -6 \\ 3 + 3\mu \end{pmatrix}$$

$$\therefore \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = -10\mu$$

$$\therefore R^2 = (-5\mu)^2 + (-6)^2 + (3 + 3\mu)^2 - \left(\frac{-10\mu}{2} \right)^2 \quad \text{(by (ii))}$$

$$= 9\mu^2 + 18\mu + 45$$

The minimum value of R^2 occurs when

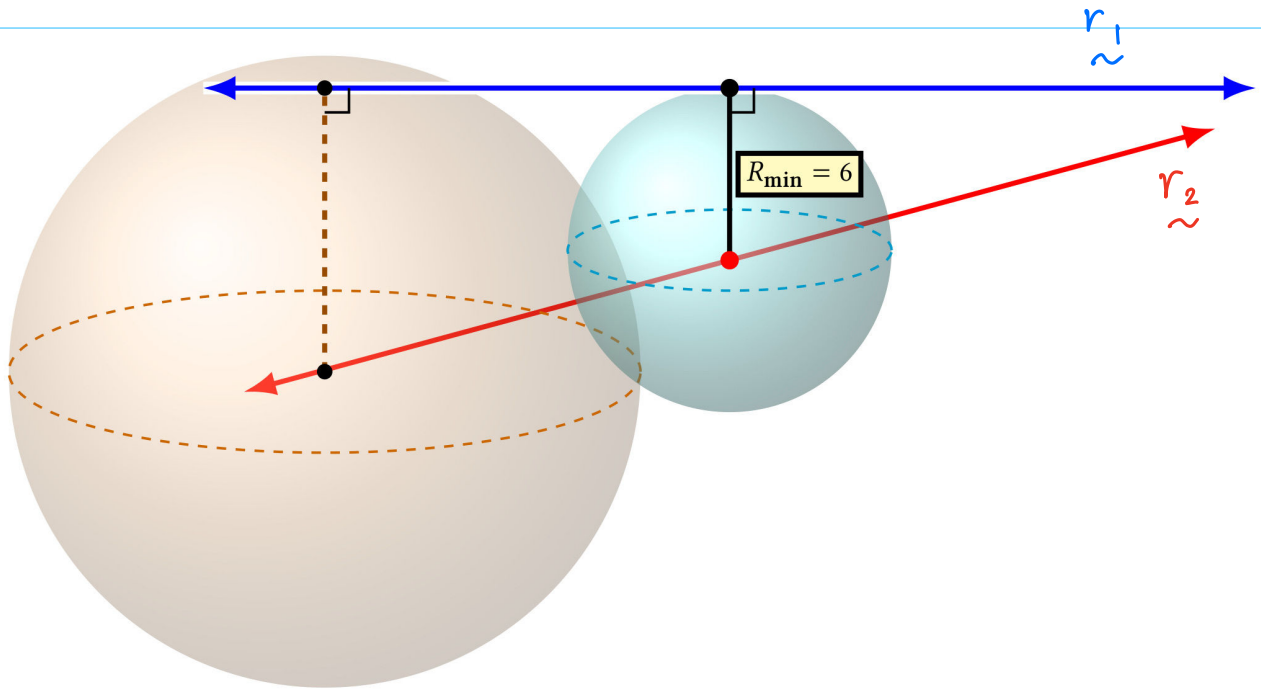
$$\mu = -\frac{18}{2 \times 9} \qquad \left(x = -\frac{b}{2a}\right)$$

$$= -1$$

$\therefore$ min $R^2 = 9(-1)^2 + 18(-1) + 45$

$= 36$

$\therefore$ min distance $= 6$ ✓



Marker's Comments

- Part (d) was generally not well done. Students are advised to read the above solutions and reattempt this question until mastery, particularly part (iv).

Question 16(a)(i)	Marks
· Provides correct solution.	3 marks
· Shows a_0 is prime.	2 marks
· Shows $P(a_0)$ is divisible by a_0 and considers factors of prime.	1 mark

Solution

$$\begin{aligned} P(a_0) &= a_n a_0^n + a_{n-1} a_0^{n-1} + \cdots + a_1 a_0 + a_0 \\ &= a_0(a_n a_0^{n-1} + a_{n-1} a_0^{n-2} + \cdots + a_1 + 1), \end{aligned}$$

so $P(a_0)$ is divisible by a_0.

However, $P(a_0)$ is prime, since $P(x)$ is prime for every integer input.
The only factors of a prime are 1 and itself, and a_0 is a factor of the prime $P(a_0)$.

It follows that either $a_0 = 1$ or $a_0 = P(a_0)$. ✓
However, since $P(0) = a_0$, it follows that a_0 is prime, since $P(x)$ is prime for every integer input. ✓

So we must have $P(a_0) = a_0$ as required. ✓

Comments

• Solutions without explanations received 0 marks.
• Students who managed to show $a_0 = 1$ or $a_0 = P(a_0)$ but did not show $a_0 \neq 1$ were given 2 marks out of 3.

Question 16(a)(ii)	Marks
· Provides correct solution.	1 mark

Solution

From (i) we know that $P(a_0) = a_0$. Then, since $a_0 > 0$ (as it is prime),

$$\begin{aligned} a_n a_0^n + a_{n-1} a_0^{n-1} + \cdots + a_1 a_0 + a_0 &= a_0 \\ a_0(a_n a_0^{n-1} + a_{n-1} a_0^{n-2} + \cdots + a_1 + 1) &= a_0 \\ a_n a_0^{n-1} + a_{n-1} a_0^{n-2} + \cdots + a_1 + 1 &= 1 \\ a_n a_0^{n-1} + a_{n-1} a_0^{n-2} + \cdots + a_1 &= 0. \end{aligned}$$

The only way the last equality can hold is if $a_n = a_{n-1} = a_{n-2} = \cdots = a_1 = 0$, because $a_1, a_2, \ldots, a_n$ are non-negative integers and $a_0 > 0$.

Hence, $P(x) = a_0$ for all real x as required. ✓

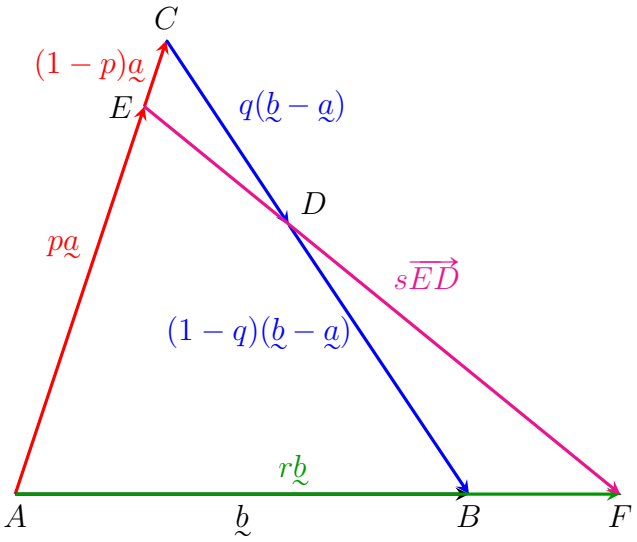
[To give an optional explicit argument, suppose for the sake of contradiction that $a_i > 0$ for some $i = 1, 2, \ldots, n$. Then LHS $= \sum_{j=1}^{n} a_j a_0^{j-1} \geqslant a_i a_0^{i-1} > 0 =$ RHS, since $a_i > 0$ and $a_0 > 0$. Thus LHS $>$ RHS, a contradiction.]

Comments

• This was a challenging question requiring insight. Only one student solved this part.

Question 16(b)	**Marks**
· Provides correct solution.	4 marks
· Simplifies $\frac{r}{r-1}$.	3 marks
· Equates coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	2 marks
· Introduces relevant scalars.	1 mark

Solution



Introduce four scalars $p, q, r, s \in \mathbb{R}^+$ satisfying

$$\begin{aligned}\overrightarrow{AE} &= p\underset{\sim}{a}\\ \overrightarrow{CD} &= q(\underset{\sim}{b}-\underset{\sim}{a})\\ \overrightarrow{AF} &= r\underset{\sim}{b}\\ \overrightarrow{AE}+s\overrightarrow{ED} &= \overrightarrow{AF}\ (*)\ \checkmark\end{aligned}$$

Using $\overrightarrow{ED} = \overrightarrow{EC} + \overrightarrow{CD}$, equation $(*)$ can be written as

$$p\underset{\sim}{a} + s\left[(1-p)\underset{\sim}{a} + q(\underset{\sim}{b}-\underset{\sim}{a})\right] = r\underset{\sim}{b}.$$

Equate coefficients of $\underset{\sim}{a}$:

$$\begin{aligned}p + s(1-p) - sq &= 0\\ s(1-p-q) &= -p\\ s &= \frac{p}{p+q-1}\end{aligned}$$

Equate coefficients of $\underset{\sim}{b}$:

$$\begin{aligned}sq &= r\\ r &= \frac{pq}{p+q-1}\ \checkmark\end{aligned}$$

Now calculate

$$\begin{aligned}\frac{r}{r-1} &= \frac{\dfrac{pq}{p+q-1}}{\dfrac{pq}{p+q-1}-1} \\ &= \frac{pq}{pq-p-q+1} \\ &= \frac{pq}{(p-1)(q-1)} \checkmark\end{aligned}$$

Now we have

$$\begin{aligned}\frac{\left|\overrightarrow{AF}\right|}{\left|\overrightarrow{BF}\right|} \times \frac{\left|\overrightarrow{DB}\right|}{\left|\overrightarrow{CD}\right|} \times \frac{\left|\overrightarrow{EC}\right|}{\left|\overrightarrow{AE}\right|} &= \frac{|r\underset{\sim}{b}|}{|(r-1)\underset{\sim}{b}|} \times \frac{|(1-q)(\underset{\sim}{b}-\underset{\sim}{a})|}{|q(\underset{\sim}{b}-\underset{\sim}{a})|} \times \frac{|(1-p)\underset{\sim}{a}|}{|p\underset{\sim}{a}|} \\ &= \left|\frac{r}{r-1}\right| \times \frac{\cancel{|\underset{\sim}{b}|}}{\cancel{|\underset{\sim}{b}|}} \times \frac{|1-q|}{|q|} \times \frac{\cancel{|\underset{\sim}{b}-\underset{\sim}{a}|}}{\cancel{|\underset{\sim}{b}-\underset{\sim}{a}|}} \times \frac{|1-p|}{|p|} \times \frac{\cancel{|\underset{\sim}{a}|}}{\cancel{|\underset{\sim}{a}|}} \\ &= \frac{|pq|}{|(p-1)(q-1)|} \times \frac{|1-q|}{|q|} \times \frac{|1-p|}{|p|} \\ &= \frac{\cancel{|p|}\cancel{|q|}}{\cancel{|p-1|}\cancel{|q-1|}} \times \frac{\cancel{|1-q|}}{\cancel{|q|}} \times \frac{\cancel{|1-p|}}{\cancel{|p|}} \\ &= 1 \checkmark\end{aligned}$$

Comments

- A more elaborate marking scheme is explained below:
- (1 mark) Introduces 3 scalars, no more progress.
- (1 mark) Introduces 3 scalars, subs in incorrectly into given ratio.
- (2 marks) Introduces 3 scalars, correctly substitutes into given ratio.
- (3 marks) Introduces 4 scalars, equates coefficients correctly, no more progress.
- (3 marks) Introduces 4 scalars, equates coefficients, subs into given ratio with one error.

Question 16(c)(i)	**Marks**
· Provides correct solution.	3 marks
· Shows $\dfrac{n^n}{n!} \geqslant 2^{n-1}$, for $n \geqslant 2$.	2 marks
· Substitutes $x_i = i$ for $i = 1, 2, \ldots, n$ and simplifies.	1 mark

Solution

Let $x_1 = 1, x_2 = 2, \ldots, x_n = n$ in the given AM−GM inequality:

$$\begin{aligned}
\left(\frac{1+2+\cdots+n}{n}\right)^n &\geqslant 1 \times 2 \times \cdots \times n \\
\left(\frac{\cancel{n}(n+1)}{2\cancel{n}}\right)^n &\geqslant n! \quad \text{(using sum of an arithmetic series)} \\
\frac{(n+1)^n}{2^n} &\geqslant n!, \quad \text{for } n \geqslant 1 \checkmark
\end{aligned}$$

Now replace every appearance of n with $n-1$, carefully noting the change in domain:

$$\begin{aligned}
\frac{n^{n-1}}{(n-1)!} &\geqslant 2^{n-1}, \quad \text{for } n-1 \geqslant 1 \\
\frac{n}{n} \times \frac{n^{n-1}}{(n-1)!} &\geqslant 2^{n-1}, \quad \text{for } n \geqslant 2 \\
\frac{n^n}{n!} &\geqslant 2^{n-1}, \quad \text{for } n \geqslant 2 \checkmark
\end{aligned}$$

Finally, substitute $n = 1$ to verify that the inequality holds for $n = 1$:

$$\begin{aligned}
\text{LHS} &= \frac{1^1}{1!} \\
&= 1 \\
&= 2^0 \\
&= \text{RHS}
\end{aligned}$$

Therefore, $\dfrac{n^n}{n!} \geqslant 2^{n-1}$ for $n \geqslant 1$ as required. ✓

Comments

• Students received no marks for proof by mathematical induction. This is because students were required to prove $\left(\dfrac{k+1}{k}\right)^k \geqslant 2$ in the inductive step, which itself is a difficult statement requiring a second induction to prove (or a monotonicity argument).

Question 16(c)(ii)	**Marks**
· Provides correct solution.	2 marks
· Shows that $f(x) = \dfrac{x^x}{a!}$ for $x \in [a, a+1)$.	1 mark

Solution

Note that if $a \leqslant x < a+1$ then $\lfloor x \rfloor = a$ by definition.
Hence, for $x \in [a, a+1)$,

$$\begin{aligned} f(x) &= \frac{x^x}{\lfloor x \rfloor !} \\ &= \frac{x^x}{a!} \checkmark \end{aligned}$$

To show $f(x)$ is increasing for $x \in [a, a+1)$, it suffices to show $f'(x) > 0$ for $x \in [a, a+1)$.

$$\begin{aligned} f(x) &= \frac{x^x}{a!} \\ &= \frac{e^{\ln(x^x)}}{a!} \\ &= \frac{e^{x \ln x}}{a!} \\ \therefore f'(x) &= \frac{(\ln x + 1) e^{x \ln x}}{a!}. \end{aligned}$$

Note that since $x \geqslant 1$ we have $\ln x \geqslant 0$, so that $\ln x + 1 \geqslant 1 > 0$.
Also, $e^t > 0$ for all real t, so $e^{x \ln x} > 0$. Finally $a!$ is positive, since a is a positive integer.
Therefore $f'(x) > 0$, as required.✓

Comments

• Students did not notice that $\lfloor x \rfloor = a$ for $a \leqslant x < a+1$.
• Even so, a small number of students were able to obtain 1 mark by using the quotient rule (and using $\lfloor x \rfloor$ is constant). Their expression looked like this:

$$f'(x) = \frac{(\ln x + 1) e^{x \ln x} \lfloor x \rfloor ! + e^{x \ln x} \dfrac{d}{dx} \lfloor x \rfloor !}{\lfloor x \rfloor !^2}$$

Question 16(c)(iii)	**Marks**
· Provides correct solution.	2 marks
· Uses (ii) to show $\displaystyle\int_1^{n+1} \frac{x^x}{\lfloor x \rfloor!}\,dx \geqslant \sum_{i=1}^{n} f(i)$	1 mark

Solution

From (ii) we know that $f(x) \geqslant f(a)$ for $x \in [a, a+1)$, since $f(x)$ is increasing there.

$$\begin{aligned}
\int_1^{n+1} \frac{x^x}{\lfloor x \rfloor!}\,dx &= \int_1^2 \frac{x^x}{\lfloor x \rfloor!}\,dx + \int_2^3 \frac{x^x}{\lfloor x \rfloor!}\,dx + \cdots + \int_n^{n+1} \frac{x^x}{\lfloor x \rfloor!}\,dx \\
&\geqslant \int_1^2 f(1)\,dx + \int_2^3 f(2)\,dx + \cdots + \int_n^{n+1} f(n)\,dx \\
&= f(1) + f(2) + \cdots + f(n) \checkmark \\
&\geqslant \frac{1^1}{\lfloor 1 \rfloor!} + \frac{2^2}{\lfloor 2 \rfloor!} + \cdots + \frac{n^n}{\lfloor n \rfloor!} \\
&\geqslant \frac{1^1}{1!} + \frac{2^2}{2!} + \cdots + \frac{n^n}{n!} \quad (\text{since } \lfloor k \rfloor = k \text{ for integers } k.) \\
&\geqslant 2^{1-1} + 2^{2-1} + \cdots + 2^{n-1} \quad (\text{from (i)}) \\
&= 2^0 + 2^1 + \cdots + 2^{n-1} \\
&= \frac{2^n - 1}{2 - 1} \quad (\text{using sum of a geometric series}) \\
&= 2^n - 1 \checkmark
\end{aligned}$$

Comments

• Unfortunately, no marks were awarded in (c)(iii) due to a common misconception. Students assumed that $\dfrac{x^x}{\lfloor x \rfloor!} \geqslant 2^{x-1}$ for all real x. However, from (c)(i), we only know this inequality holds for *positive integers*. It was necessary to use the parts (c)(i) and (c)(ii) together to solve this problem, as detailed above.