SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2026 Trial Examination

Form VI Mathematics Extension 2

Wednesday 12th August, 2026
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Sixteen Questions — 100 Marks

Section I (10 marks) Questions 1 – 10

- This section is multiple choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 16

- Each question is worth 15 marks.
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple-choice answer sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.

Checklist

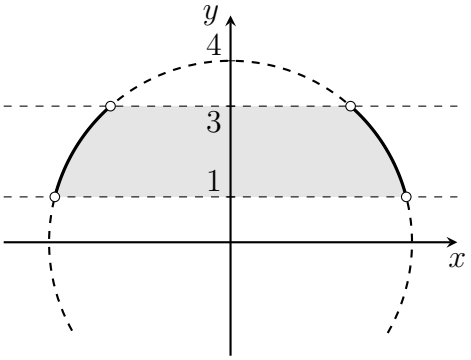
- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 70 pupils

Writer: WJM

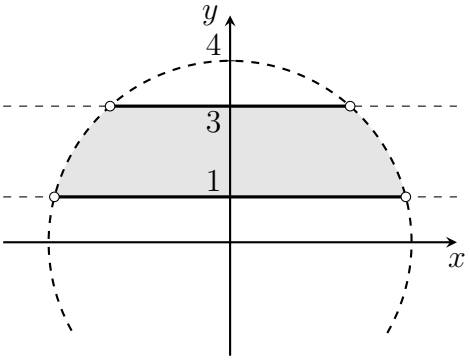
Section I

1. Let R be the region in the complex plane defined by $1 \leq \text{Im}(z) \leq 3$ and $|z| < 4$. **1**
Which diagram best represents the region R?

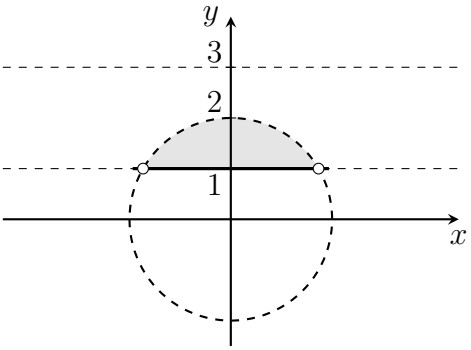
(A)



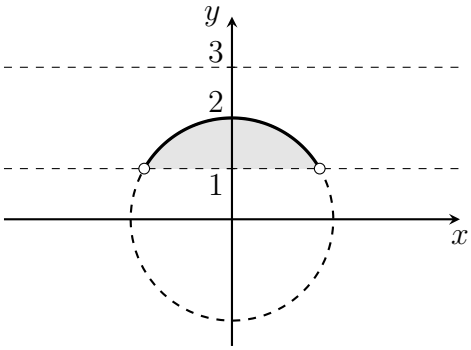
(B)



(C)



(D)



2. A driving manual contains the following statement: **1**

'If someone is bad at driving, then they will never miss an exit.'

Which of the following is logically equivalent to this statement?

(A) If someone is good at driving, then they will never miss an exit.

(B) If someone is not bad at driving, then they will miss an exit.

(C) If someone misses an exit, then they are bad at driving.

(D) If someone misses an exit, then they are not bad at driving.

3. Which of the following is a correct expression for $\displaystyle\int \frac{dx}{\sqrt{5-4x-x^2}}$? **1**

(A) $\sin^{-1}\left(\dfrac{x-2}{3}\right)+c$

(B) $\sin^{-1}\left(\dfrac{x+2}{3}\right)+c$

(C) $\sin^{-1}\left(\dfrac{x-2}{9}\right)+c$

(D) $\sin^{-1}\left(\dfrac{x+2}{9}\right)+c$

4. A complex number $z=x+iy$ is defined such that $\text{Re}\left(\dfrac{1}{z}\right)=\dfrac{1}{2}$. **1**

Which of the following circles in the complex plane must z always lie on?

(A) $(x-1)^2+y^2=1$

(B) $x^2+(y-1)^2=1$

(C) $(x+1)^2+y^2=1$

(D) $x^2+(y+1)^2=1$

5. The vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are given by $\underset{\sim}{p}=\begin{pmatrix}\sin\theta\\ \cos\theta\\ \tan\theta\end{pmatrix}$ and $\underset{\sim}{q}=\begin{pmatrix}\text{cosec}\,\theta\\ \sec\theta\\ \cot\theta\end{pmatrix}$, where $0\leq\theta\leq 2\pi$. **1**

For how many values of θ is $\text{proj}_{\underset{\sim}{p}}\,\underset{\sim}{q}$ a unit vector?

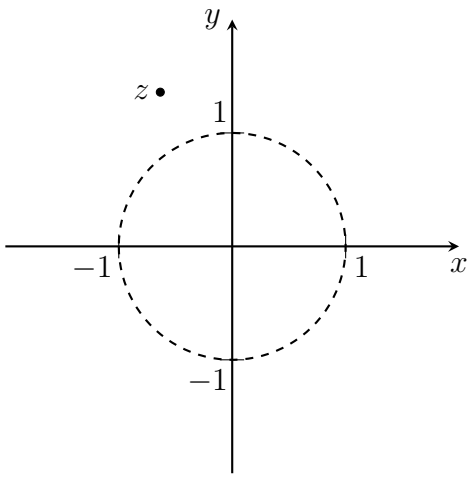
(A) 0

(B) 1

(C) 2

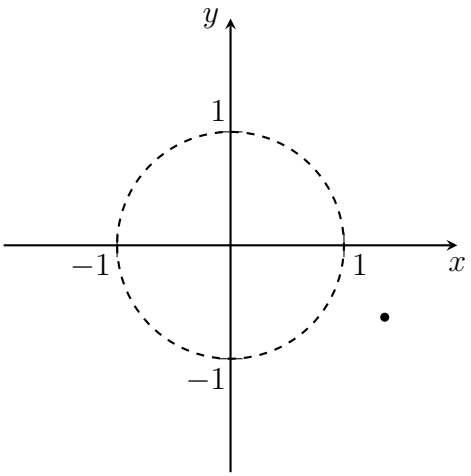
(D) 4

6. The diagram below shows the complex number $z = re^{i\theta}$ on the Argand diagram. **1**

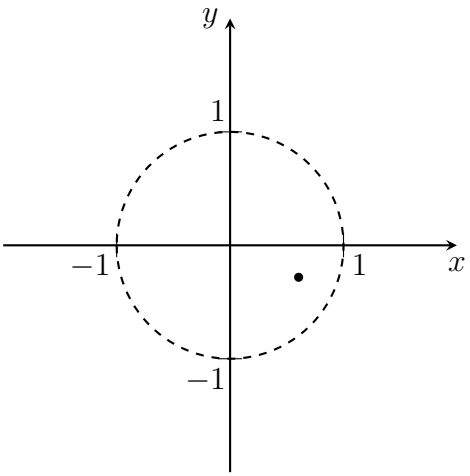


Which of the following diagrams best shows the position of $\dfrac{ir^2}{z}$?

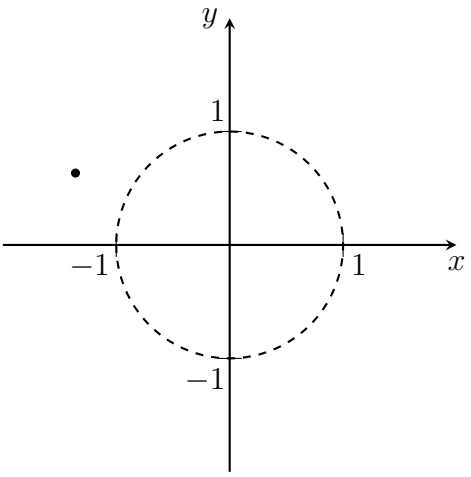
(A)



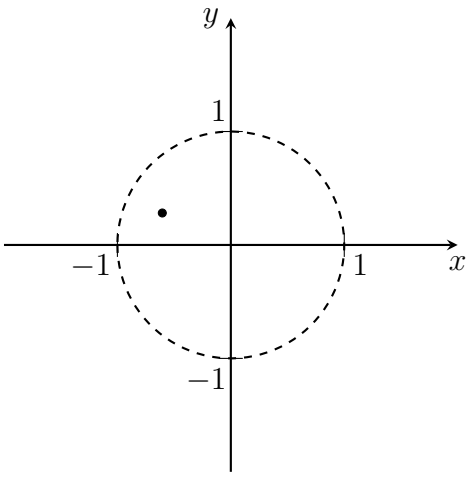
(B)



(C)



(D)



7. Consider the statement: **1**

$$\forall\, p, q \in \mathbb{Z}^+, \exists\, m, n \in \mathbb{Z}^+ \text{ such that } \sqrt{p+q} = \sqrt{m} + \sqrt{n}.$$

Which of the following could be used as a counterexample to disprove this statement?

(A) $p = 2, q = -2$

(B) $p = 6, q = 3$

(C) $p = 10, q = 8$

(D) $p = 2, q = 4$

8. It is known that the vector $\underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ is perpendicular to $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$, where a, b, $c \neq 0$. Which of the following vectors is NOT perpendicular to $\underset{\sim}{r}$? **1**

(A) $(a+c)\underset{\sim}{i} + (b+c)\underset{\sim}{j} + c\underset{\sim}{k}$

(B) $b\underset{\sim}{i} + a\underset{\sim}{j} - c\underset{\sim}{k}$

(C) $b\underset{\sim}{i} + a\underset{\sim}{j} + (a-b+c)\underset{\sim}{k}$

(D) $(2c-a)\underset{\sim}{i} + (b-2c)\underset{\sim}{j} + (a+b-c)\underset{\sim}{k}$

9. The complex number z is defined such that $-\frac{\pi}{4} \leq \text{Arg}(z-i) \leq \frac{\pi}{4}$. **1**

Which of the following describes the set of values that $\text{Arg}(z^2+1)$ can take?

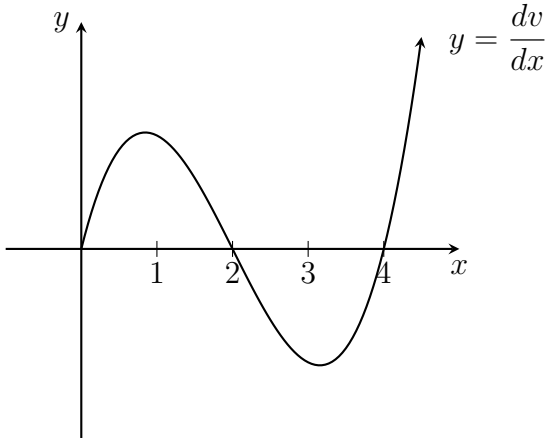
(A) $-\frac{\pi}{2} < \text{Arg}(z^2+1) < \frac{\pi}{2}$

(B) $-\frac{\pi}{2} < \text{Arg}(z^2+1) < \frac{3\pi}{4}$

(C) $-\frac{3\pi}{4} < \text{Arg}(z^2+1) < \frac{\pi}{4}$

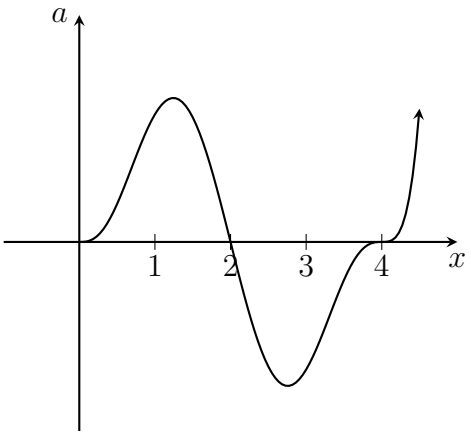
(D) $-\frac{\pi}{4} < \text{Arg}(z^2+1) < \pi$

10. The graph below shows the derivative of the velocity v of a particle as a function of its displacement x, where $x \geq 0$. **1**

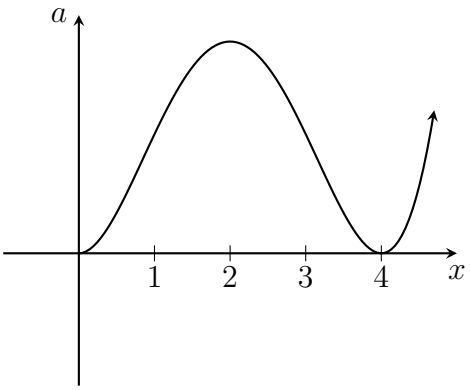


It is known that if the particle is at $x = 0$, it will be stationary and will not move. Which of the following graphs best shows the acceleration of the particle as a function of its displacement?

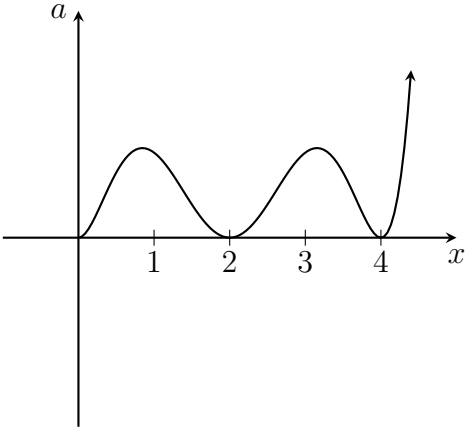
(A)



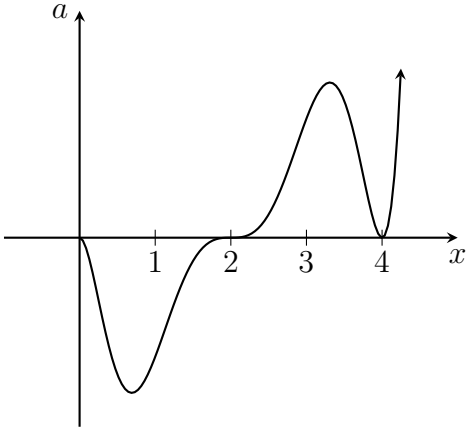
(B)



(C)



(D)



End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet.

(a) Solve $z^2 - 4z + 8 = 0$. **2**

(b) The complex numbers w and z are given by $w = 8e^{\frac{3i\pi}{4}}$ and $z = 2e^{\frac{i\pi}{6}}$. **2**

Find the modulus and principal argument of $\dfrac{w}{z}$.

(c) Find the following:

(i) $\displaystyle\int (\tan^2 x + 1)\sec^2 x \, dx$ **2**

(ii) $\displaystyle\int \frac{x^2}{4 + x^2}\, dx$ **2**

(iii) $\displaystyle\int x \cos x \, dx$ **2**

(d) Two lines are given by $\underset{\sim}{r}_1 = \begin{pmatrix} 1 \\ -2 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{r}_2 = \begin{pmatrix} -6 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$,

where $\lambda, \mu \in \mathbb{R}$.

(i) Show that the lines intersect, and find their point of intersection. **3**

(ii) Determine the acute angle between the lines, giving your answer correct to the nearest degree. **2**

The paper continues on the next page

QUESTION TWELVE (15 marks) Start a new answer booklet.

(a) (i) Show that if n is an integer, then $n^2 + 7n + 6$ is even. **2**

(ii) Hence determine, with justification, the number of integer solutions to the equation $(n^2 + 7n + 6)^7 - 16n^5 = 3$. **1**

(b) (i) Find the values of A, B, and C such that **2**

$$\frac{3x^2 - 6x + 13}{(x-5)(x^2+4)} = \frac{A}{x-5} + \frac{Bx + C}{x^2+4}.$$

(ii) Hence find $\displaystyle\int \frac{3x^2 - 6x + 13}{(x-5)(x^2+4)}\,dx$. **1**

(c) An object moving in simple harmonic motion has displacement x m and velocity $v\,\text{ms}^{-1}$, and its motion satisfies $v^2 = 27 - 18x - 9x^2$.

(i) Find the centre of motion and the amplitude. **2**

(ii) Determine the period and hence find the distance that the particle will travel in 30π seconds. **2**

(iii) If the object has a mass of 5 kg, find the maximum force that acts on the object throughout its motion. **1**

(d) The equation $x^3 + bx^2 + cx + 3b = 0$, where b and c are real, has a complex root $b + ic$. Using the relationship between the roots and the coefficients:

(i) show that the equation has a real root of $-3b$, **1**

(ii) find the possible values of b and c. **3**

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QUESTION THIRTEEN (15 marks) Start a new answer booklet.

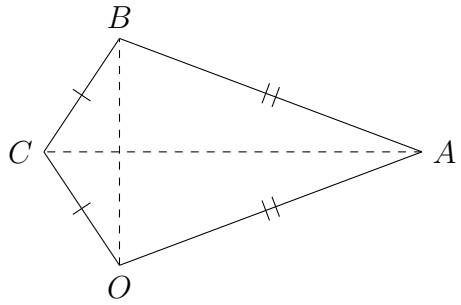
(a) Consider the statement: 'If n is a multiple of 6, then $\left(1-i\sqrt{3}\right)^n$ is real'.

(i) Prove that the statement is true. **2**

(ii) State the converse and determine if it is true, justifying your answer. **2**

(b) Use integration by parts to find $\int e^x \cos 2x \, dx$. **3**

(c) The diagram below shows the quadrilateral $OABC$, where $OA = BA$ and $OC = BC$.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

(i) Use the fact that $|\underset{\sim}{a}| = |\underset{\sim}{b} - \underset{\sim}{a}|$ to show that $\underset{\sim}{b} \cdot \underset{\sim}{b} = 2\underset{\sim}{a} \cdot \underset{\sim}{b}$. **1**

(ii) Use vector methods to show that $OB \perp AC$. **2**

(d) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^4$, show that **3**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

(ii) Hence find the roots of the polynomial $x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$. **2**

The paper continues on the next page

QUESTION FOURTEEN (15 marks) Start a new answer booklet.

(a) On a distant planet, an astronaut determines that a particular object of mass m kg experiences a resistive force of mkv^2 when it moves through the atmosphere, where v is the velocity of the object and k is a positive constant. The magnitude of acceleration due to gravity on the planet is $G\,\text{ms}^{-2}$.

When falling, the equation of motion of the particle is $\ddot{x} = -G + kv^2$, and the astronaut determines that the object has a terminal velocity of $40\,\text{ms}^{-1}$ in the negative direction.

(i) Show that $\sqrt{\dfrac{G}{k}} = 40$. **1**

(ii) The object is projected upwards with a velocity of $40\,\text{ms}^{-1}$. When it is moving upwards, its equation of motion is $\ddot{x} = -G - kv^2$.

(α) Show that $v = 40\tan\left(\frac{\pi}{4} - 40kt\right)$. **2**

(β) The astronaut observes that it takes 4 seconds for it to reach its maximum height. Calculate the exact value of G. **2**

(b) Two complex numbers z_1 and z_2 lie on the unit circle, and are given by $z_1 = e^{i\alpha}$ and $z_2 = e^{i\beta}$ where $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq \pi$.

(i) Show that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. **1**

(ii) The complex number z_3 is defined by $z_3 = \dfrac{1}{z_1 + z_2}$. **3**

For certain values of α and β, it is found that z_3 has modulus $\dfrac{1}{\sqrt{3}}$.

Using the fact that

$$e^{i\alpha} + e^{i\beta} = e^{i(\alpha+\beta)/2}\left(e^{i(\alpha-\beta)/2} + e^{-i(\alpha-\beta)/2}\right),$$

or otherwise, find the possible values of $\alpha - \beta$.

(c) (i) Use mathematical induction to show that $x + 1$ is a factor of $x^n + 1$ for all **odd** positive integers n. **2**

(ii) It is known that for some n, k, $m \in \mathbb{Z}^+$, $n = m \times 2^k$. **2**

Use proof by contraposition to show that if n is not a power of 2, then m has an odd factor larger than 1.

(iii) Deduce that if $2^n + 1$ is prime, then n is a power of 2. **2**

The paper continues on the next page

QUESTION FIFTEEN (15 marks) Start a new answer booklet.

(a) Let $I_n = \displaystyle\int_0^1 \left(1 - \sqrt{x}\right)^n dx$ for integers $n \geq 0$.

(i) Use integration by parts to show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$. **3**

(ii) Hence show that $I_n = \dfrac{2}{(n+1)(n+2)}$. **1**

(iii) By considering the partial fractions decomposition of I_n, or otherwise, deduce that $I_0 + I_1 + I_2 + \cdots + I_n < 2$. **2**

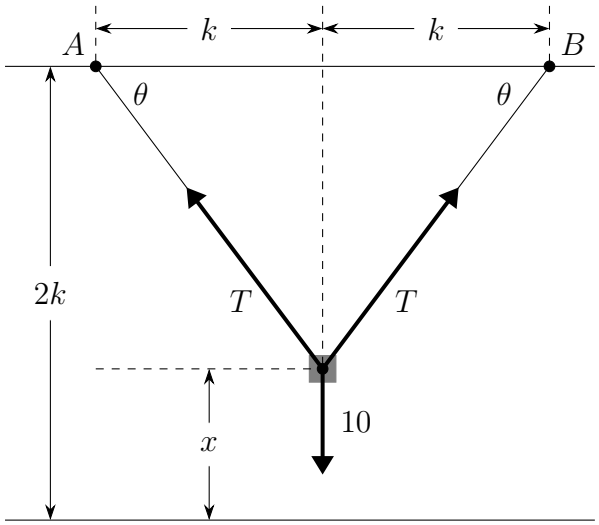
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QUESTION FIFTEEN (Continued)

(b) Given that p and q are constants, use integration by substitution to show that **1**

$$\int \frac{p-x}{\sqrt{q^2+(p-x)^2}}\,dx = -\sqrt{q^2+(p-x)^2}+c,$$ where c is a constant.

(c) The diagram below shows an object of mass 1 kg being suspended from the ceiling by two cables of equal length, which are attached to motors at the points A and B.



The motors at A and B both pull on their respective cables, creating a tension of T newtons in each cable. The motors are designed such that the magnitude of the tension in each cable is a constant T newtons at all times. The distance between A and B is $2k$ metres, and the ceiling is $2k$ metres above the ground.

Let the height of the object above the ground be x metres, the angle between each cable and the ceiling be θ, and the magnitude of acceleration due to gravity be $10\,\text{ms}^{-2}$.

(i) Show that $\ddot{x} = \dfrac{2T(2k-x)}{\sqrt{k^2+(2k-x)^2}} - 10$. **2**

The object is initially at rest on the ground.

(ii) Using part (b), or otherwise, show that $(\dot{x})^2 = 4T\left(k\sqrt{5} - \sqrt{k^2+(2k-x)^2}\right) - 20x$. **2**

(iii) Show that if the object touches the ceiling, the minimum magnitude of the force produced by each motor is $T = 5\varphi$ newtons, where $\varphi = \dfrac{1+\sqrt{5}}{2}$. **1**

(iv) When the value of T is such that the object just touches the ceiling, the maximum velocity of the object is reached when it has travelled a fraction P of the distance between the floor and the ceiling. **3**

Given that $\varphi^2 - 1 = \varphi$, show that $P = \dfrac{2\sqrt{\varphi}-1}{2\sqrt{\varphi}}$.

The paper continues on the next page

QUESTION SIXTEEN (15 marks) Start a new answer booklet.

(a) (i) Let $f(x)$ be a function that is differentiable on the domain $x \in \left[\frac{1}{2}, 1\right]$. **2**
Use integration by parts to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(\sin x)\,dx = \tfrac{\pi}{6}\left(3f(1) - f(\tfrac{1}{2})\right) - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos x f'(\sin x)\,dx\,.$$

(ii) Hence find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2\sin x + x\cos x}{\sqrt{\sin x}}\,dx\,.$ **2**

(b) The AM-GM inequality states that for positive numbers $x_1,\ x_2,\ \ldots,\ x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}\,. \quad \text{Do not prove this.}$$

(i) Show that if $a_1,\ a_2,\ \ldots,\ a_n$ are positive numbers, and $k_1,\ k_2,\ \ldots,\ k_n$ are positive integers, then **1**

$$\frac{a_1k_1 + a_2k_2 + \cdots + a_nk_n}{k_1 + k_2 + \cdots + k_n} \geq \left(a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}\right)^{\frac{1}{k_1+k_2+\cdots+k_n}}\,.$$

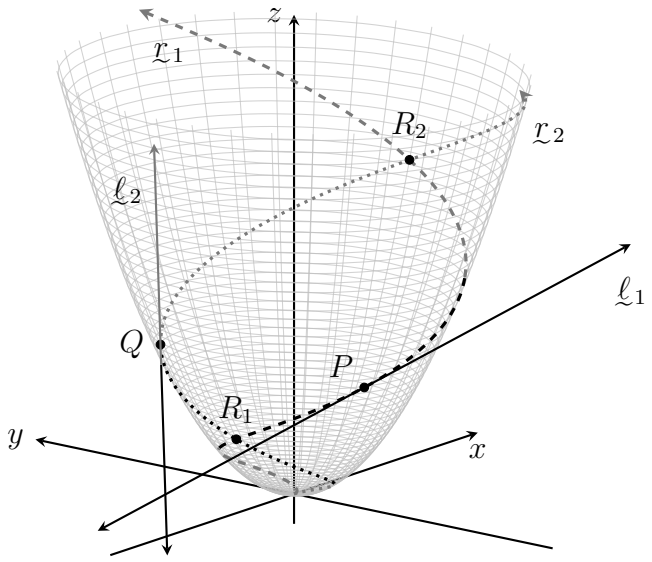
(ii) Given that the inequality in part (i) is also true when $k_1,\ k_2,\ \ldots,\ k_n$ are positive numbers, show that if $a, b, c > 0$ and $a + b + c = 1$, then **3**

$$a^ab^bc^c + a^bb^cc^a + a^cb^ac^b \leq 1\,.$$

The question continues on the next page

QUESTION SIXTEEN (Continued)

(c) The diagram below shows the *paraboloid* $x^2 + y^2 = kz$ for some constant $k > 0$, which is the surface formed when the parabola $z = \frac{1}{k}x^2$ is rotated about the z-axis.



The curve $\underset{\sim}{r}_1$, shown by the dashed curve above, spirals around the paraboloid in an anti-clockwise direction when viewed from above. The curve $\underset{\sim}{r}_2$, shown by the dotted curve above, spirals around the paraboloid in a clockwise direction.

The point P on the curve $\underset{\sim}{r}_1$ and the point Q on the curve $\underset{\sim}{r}_2$ can be parametrised by

$$P\left(p\cos p, p\sin p, \tfrac{1}{k}p^2\right) \quad \text{and} \quad Q\left(q\cos q, -q\sin q, \tfrac{1}{k}q^2\right),$$

where p and q are non-negative parameters.

Let $\underset{\sim}{\ell}_1$ be the tangent to $\underset{\sim}{r}_1$ at P and $\underset{\sim}{\ell}_2$ be the tangent to $\underset{\sim}{r}_2$ at Q with equations

$$\underset{\sim}{\ell}_1 = \begin{pmatrix} p\cos p \\ p\sin p \\ \frac{1}{k}p^2 \end{pmatrix} + \lambda \begin{pmatrix} -p\sin p + \cos p \\ p\cos p + \sin p \\ \frac{2}{k}p \end{pmatrix} \quad \text{and} \quad \underset{\sim}{\ell}_2 = \begin{pmatrix} q\cos q \\ -q\sin q \\ \frac{1}{k}q^2 \end{pmatrix} + \mu \begin{pmatrix} -q\sin q + \cos q \\ -q\cos q - \sin q \\ \frac{2}{k}q \end{pmatrix}.$$

The point R_n is the nth point of intersection of the curves $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ above the xy-plane. The points R_1 and R_2 are shown in the diagram above.

(i) Find, in terms of n, the value of the parameter p that corresponds to the point R_n. **1**

(ii) For all integers $n \geq 1$, $\overrightarrow{R_nR_{n+1}} \perp \overrightarrow{R_{n+1}R_{n+2}}$.

(α) Find the exact value of k. **3**

(β) Let θ_n be the obtuse angle between the two tangents $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ at R_n. **3**

Find $\lim\limits_{n\to\infty} \tan\left(\dfrac{\theta_n}{2}\right)$.

——— END OF PAPER ———

Maths Ext. 2 Trial 2026

Multiple Choice

(1) B

(2) Contrapositive: "If someone misses an exit, then they are not bad at driving". $\Rightarrow$ (D)

(3) $\int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

$\Rightarrow$ (B)

(4) $\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$

If $Re(\frac{1}{z}) = \frac{1}{2}$, $\frac{x}{x^2+y^2} = \frac{1}{2}$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$ $\Rightarrow$ (A)

(5) $\text{proj}_{\underset{\sim}{\hat{p}}} \underset{\sim}{q} = \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{\underset{\sim}{p} \cdot \underset{\sim}{p}} \underset{\sim}{p}$

$= \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}|^2} \times |\underset{\sim}{p}| \times \hat{\underset{\sim}{p}}$

$= \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}|} \hat{\underset{\sim}{p}}$

$= \frac{1+1+1}{\sqrt{\sin^2\theta + \cos^2\theta + \tan^2\theta}} \hat{\underset{\sim}{p}}$

$= \frac{3}{\sqrt{1+\tan^2\theta}} \hat{\underset{\sim}{p}}$

$= \frac{3}{\sqrt{\sec^2\theta}} \hat{\underset{\sim}{p}}$

$= 3|\cos\theta| \hat{\underset{\sim}{p}}$ $\therefore$ Need $|\cos\theta| = \frac{1}{3}$, $\cos\theta = \pm\frac{1}{3}$

$\Rightarrow$ 4 solutions (D)

(6) $\frac{ir^2}{z} = \frac{ir^2}{re^{i\theta}}$

$= ire^{-i\theta}$

* Conjugate of z, multiplied by i

$\Rightarrow$ (A)

(7) (D) i.e. $\exists m, n \in \mathbb{Z}^+ : \sqrt{2+4} = \sqrt{m} + \sqrt{n}$

$\sqrt{6} = \sqrt{m} + \sqrt{n}$

$6 = m + n + 2\sqrt{mn}$

Impossible for $m, n \in \mathbb{Z}^+$.

(8) $(\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}) = a - b + 2c$

$\therefore \quad a - b + 2c = 0$

A: $((a+c)\underset{\sim}{i} + (b+c)\underset{\sim}{j} + c\underset{\sim}{k}) \cdot (\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = a+c-(b+c)+2c$

$= a - b + 2c$

$= 0$

B: $(b\underset{\sim}{i} + a\underset{\sim}{j} - c\underset{\sim}{k}) \cdot (\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = b - a - 2c$

$= -(a - b + 2c)$

$= 0$

C: $(b\underset{\sim}{i} + a\underset{\sim}{j} + (a-b+c)\underset{\sim}{k}) \cdot (\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$

$= b - a + 2a - 2b + 2c$

$= a - b + 2c$

$= 0$

D: $((2c-a)\underset{\sim}{i} + (b-2c)\underset{\sim}{j} + (a+b-c)\underset{\sim}{k}) \cdot (\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$

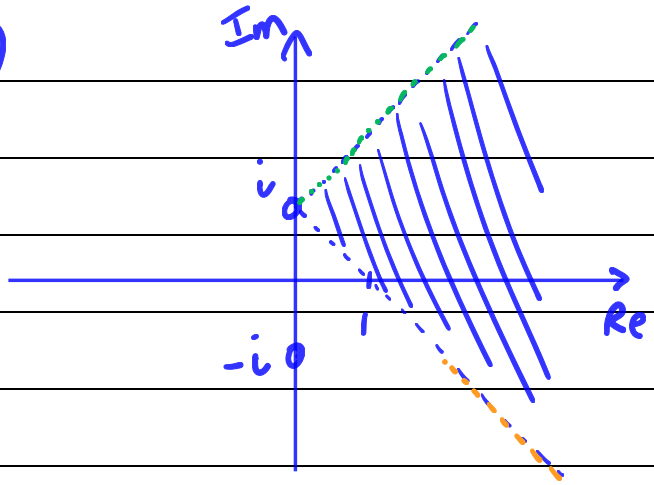
$= 2c - a + 2c - b + 2a + 2b - 2c$

$= a + b + 2c$

$\neq 0$

$\Rightarrow$ (D)

(9)



$$\arg(z^2+1) = \arg\left((z+i)(z-i)\right)$$
$$= \arg(z+i) + \arg(z-i)$$

When $\arg(z-i) = \frac{\pi}{4}$ (green branch), $\arg(z+i) < \frac{\pi}{2}$

$$\therefore \arg(z+i) + \arg(z-i) < \frac{3\pi}{4}$$

Also, when $\arg(z-i) = -\frac{\pi}{4}$ and $\arg(z+i) < 0$ (orange brach),

$$\arg(z+i) > -\frac{\pi}{4}$$

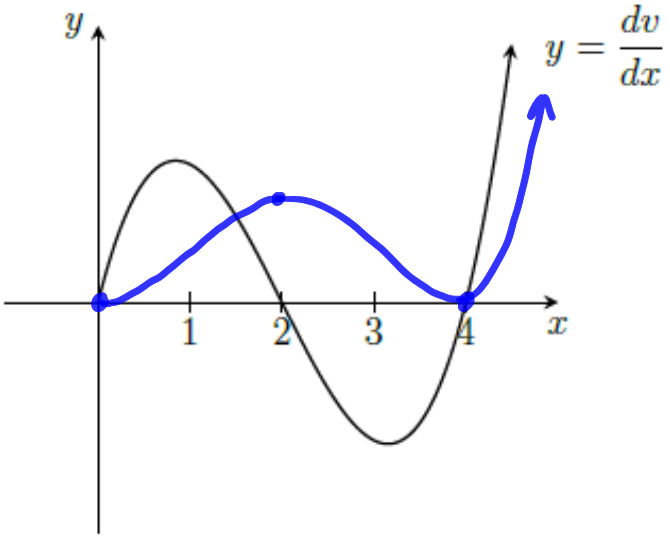
$$\therefore \arg(z+i) + \arg(z-i) > -\frac{\pi}{2}$$

i.e. $$-\frac{\pi}{2} < \arg(z^2+1) < \frac{3\pi}{4}$$

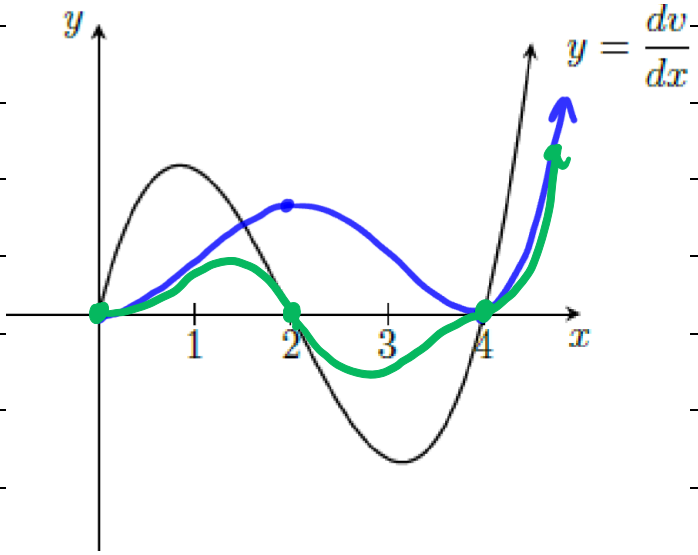
$\Rightarrow$ (B)

(10) $a = v \cdot \frac{dv}{dx}$

Taking the primitive of $\frac{dv}{dx}$ with $x = 0$, $v = 0$ gives:



Taking the product of v and $\frac{dv}{dx}$ gives:



$\Rightarrow$ (A)

Question 11

(a) $z^2 - 4z + 8 = 0$

$(z-2)^2 = -8 + 4$

$= -4$

$= (2i)^2$ ✓

$z = 2 \pm 2i$ ✓

(b) $\frac{w}{z} = \frac{8e^{3i\pi/4}}{2e^{i\pi/6}}$

$= 4e^{7i\pi/12}$

Modulus: 4 ✓

Argument: $\frac{7\pi}{12}$ ✓

(c) (i) $\int (\tan^2 x + 1)\sec^2 x \, dx$ $\quad u = \tan x$ ✓ (recognise substitution)

$= \int (u^2 + 1)\, du$ $\quad du = \sec^2 x \, dx$

$= \frac{1}{3}u^3 + u + c$

$= \frac{1}{3}\tan^3 x + \tan x + c$ ✓

(ii) $\int \frac{x^2}{4+x^2}\, dx = \int \left(\frac{x^2+4}{4+x^2} - \frac{4}{4+x^2}\right) dx$ ✓ (useful algebraic manipulation)

$= \int \left(1 - \frac{4}{4+x^2}\right) dx$

$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$ ✓

(iii) $\int x\cos x \, dx = x\sin x - \int \sin x \, dx$ ✓

$v' = \cos x \quad u = x$

$v = \sin x \quad u' = 1$

$= x\sin x + \cos x + c$ ✓

(d)(i) Solving simultaneously: $\begin{pmatrix} 1+\lambda \\ -2-\lambda \\ 14+3\lambda \end{pmatrix} = \begin{pmatrix} -6+2\mu \\ 3-\mu \\ -1+3\mu \end{pmatrix}$

Equating $\underset{\sim}{i}$: $1+\lambda = -6+2\mu$

$\lambda - 2\mu = -7$ ①

Equating $\underset{\sim}{j}$: $-2-\lambda = 3-\mu$

$\mu - \lambda = 5$ ②

✓ Forming 2 equations

① + ②: $-\mu = -2$

$\mu = 2$

$\therefore 2-\lambda = 5$

$\lambda = -3$

Check $\underset{\sim}{k}$: LHS $= 14+3\lambda$

$= 14+3\times-3$

$= 5$

RHS $= -1+3\times2$

$= 5$ ✓

$\therefore$ the lines intersect.

When $\mu = 2$, $\underset{\sim}{r} = \begin{pmatrix} -6+4 \\ 3-2 \\ -1+6 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

$\therefore$ the lines intersect at $(-2, 1, 5)$. ✓

(ii) $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2+1+9$

$= 12$

$\therefore \cos\theta = \dfrac{12}{\sqrt{1+1+9}\times\sqrt{4+1+9}}$ ✓, $\theta = \cos^{-1}\left(\dfrac{12}{\sqrt{11}\times\sqrt{14}}\right)$

$\doteq 15°$ (nearest degree) ✓

Question 12

(a) (i) $n^2 + 7n + 6 = (n+6)(n+1)$

If n is even, then $n+6$ is even, so n^2+7n+6 is even. ✓

If n is odd, then $n+1$ is even, so n^2+7n+6 is even. ✓

In either case, $n^2 + 7n + 6$ is even.

(ii) Since $n^2 + 7n + 6$ is even, $(n^2+7n+6)^7$ is even.

Also, $16 \times n^5$ is even.

$\therefore (n^2+7n+6)^7 - 16n^5$ is even, but 3 is odd.

$\therefore$ there are 0 integer solutions. ✓

(b) (i) $\dfrac{3x^2-6x+13}{(x-5)(x^2+4)} \equiv \dfrac{A}{x-5} + \dfrac{Bx+C}{x^2+4}$

$3x^2 - 6x + 13 \equiv A(x^2+4) + (Bx+C)(x-5)$

$x = 5$: $58 = 29A$

$A = 2$

$x = 0$: $13 = 4A - 5C$

$13 = 8 - 5C$

$C = -1$

✓ (any one correct values)

Equate coef. of x^2: $A + B = 3$

$B = 1$ ✓ (all correct values)

(ii) $\displaystyle\int \frac{3x^2-6x+13}{(x-5)(x^2+4)}\,dx = \int \left(\frac{2}{x-5} + \frac{x-1}{x^2+4}\right) dx$

$\displaystyle= \int \left(\frac{2}{x-5} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right) dx$

$= 2\ln|x-5| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

✓ (all correct)

(c) (i) $v^2 = 27 - 18x - 9x^2$

When $v = 0$: $9x^2 + 18x - 27 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$ ✓

∴ Centre of motion: $x = \frac{-3+1}{2}$

$= -1$ ✓ both

amplitude = 2 metres.

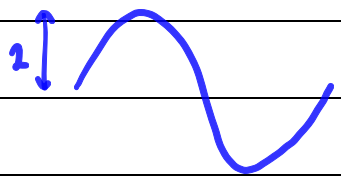
(ii) $v^2 = -9(x^2 + 2x - 3)$

$= -9((x+1)^2 - 4)$

$\Rightarrow n^2 = 9, \quad n = 3$

∴ Period = $\frac{2\pi}{3}$ ✓

In 30π seconds, particle completes $\frac{30\pi}{2\pi/3} = 45$ full oscillations.



particle travels 8m / oscillation

∴ distance travelled: $8 \times 45 = 360$ metres ✓

(iii) $\ddot{x} = -n^2(x - c)$

$= -9(x + 1)$

Max. force acts when furthest from centre of motion.

When $x = -3$, $\ddot{x} = -9 \times -2$

$= 18 \text{ ms}^{-2}$

∴ Max force = 5×18

$= 90 \text{ N}$ ✓

(d) (i) As the coefficients are real, $b-ic$ is also a root.

Let the roots be $b-ic$, $b+ic$, α

$$(b-ic)+(b+ic)+\alpha = -b$$

$$\alpha = -3b$$ ✓

(ii) $\sum \alpha\beta$: $-3b(b-ic)-3b(b+ic)+(b-ic)(b+ic) = c$

$$-3b^2+3ibc-3b^2-3ibc+b^2+c^2 = c$$

$$c^2-5b^2 = c \quad (1)$$

$\sum \alpha\beta\gamma$: $-3b(b-ic)(b+ic) = -3b$

$$b^2+c^2 = 1$$

$$b^2 = 1-c^2 \quad (2)$$ ✓

sub. (2) into (1): $c^2-5(1-c^2) = c$

$$6c^2-c-5 = 0$$

$$(c-1)(6c+5) = 0$$

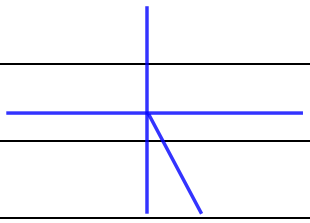
$$c = 1 \text{ or } c = -\tfrac{5}{6}$$ ✓

If $c = 1$, $b = 0$

If $c = -\frac{5}{6}$, $b^2 = \frac{11}{36}$, so $b = \pm\frac{\sqrt{11}}{6}$ ✓

Question 13

(a) (i)



$1 - i\sqrt{3} = 2\text{cis}(-\frac{\pi}{3})$

$\therefore (1-i\sqrt{3})^n = (2\text{cis}(-\frac{\pi}{3}))^n$

$= 2^n \text{cis}(-\frac{n\pi}{3})$ ✓

If n is a multiple of 6, $\exists\, m \in \mathbb{Z}$ such that $n = 6m$.

$2^n \text{cis}(-\frac{n\pi}{3}) = 2^{6m}\text{cis}(-\frac{6m\pi}{3})$

$= 2^{6m}\text{cis}(-2m\pi)$

$= 2^{6m} \in \mathbb{R}$. ✓

(ii) Converse: "If $(1-i\sqrt{3})^n$ is real, then n is a multiple of 6." ✓ (correctly identify converse)

$(1-i\sqrt{3})^n = 2^n\text{cis}(-\frac{n\pi}{3})$, which is real if $-\frac{n\pi}{3}$ is an integer multiple of π.

This is true when n is a multiple of 3, ✓ (+ conclusion) so there are values of n $(3, 9, 15, \ldots)$ where n is not a multiple of 6. $\therefore$ Converse is not true.

(b) $\int e^x \cos 2x\, dx = e^x\cos 2x + 2\int e^x \sin 2x$ ✓

$v' = e^x \quad u = \cos 2x$

$v = e^x \quad u' = -2\sin 2x$

$v' = e^x \quad u = \sin 2x$

$v = e^x \quad u' = 2\cos 2x$

$= e^x\cos 2x + 2\left(e^x \sin 2x - 2\int e^x \cos 2x\right)$ ✓

$\therefore I = e^x\cos 2x + 2e^x\sin 2x - 4I$

$5I = e^x\cos 2x + 2e^x\sin 2x + c_1$

$I = \frac{1}{5}(e^x\cos 2x + 2e^x\sin 2x) + c_2$ ✓

(c) (i) $|\underset{\sim}{a}| = |\underset{\sim}{b} - \underset{\sim}{a}|$

$\therefore \quad \underset{\sim}{a} \cdot \underset{\sim}{a} = (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b} - 2\,\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$\therefore \quad \underset{\sim}{b} \cdot \underset{\sim}{b} = 2\,\underset{\sim}{a} \cdot \underset{\sim}{b}$ ✓

(ii) From (i): $\underset{\sim}{b} \cdot \underset{\sim}{b} = 2\,\underset{\sim}{a} \cdot \underset{\sim}{b}$

Similarly, $|\underset{\sim}{c}| = |\underset{\sim}{b} - \underset{\sim}{c}|$ gives $\underset{\sim}{b} \cdot \underset{\sim}{b} = 2\,\underset{\sim}{b} \cdot \underset{\sim}{c}$ ✓

$\overrightarrow{OB} \cdot \overrightarrow{AC} = \underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a})$

$= \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a}$

$= \frac{1}{2}\,\underset{\sim}{b} \cdot \underset{\sim}{b} - \frac{1}{2}\,\underset{\sim}{b} \cdot \underset{\sim}{b}$ from above

$= 0$ ✓

$\therefore \quad \overrightarrow{OB} \perp \overrightarrow{AC}$

(d) (i) $(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$ ✓ (correct expansion)

$= \underbrace{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}_{\cos 4\theta} + i\underbrace{(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)}_{\sin 4\theta}$ ✓ by De Moivre's

$\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \div \dfrac{\cos^4\theta}{\cos^4\theta}$ ✓

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

Must show some understanding of how to reach last line

(ii) Let $\tan 4\theta = -1$: ✓

$-1 = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

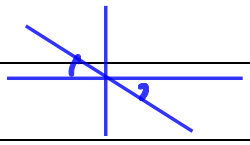
$$-1 + 6\tan^2\theta - \tan^4\theta = 4\tan\theta - 4\tan^3\theta$$

$$\tan^4\theta - 4\tan^3\theta - 6\tan^2\theta + 4\tan\theta + 1 = 0$$

Letting $x = \tan\theta$:

$$x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$$

$\therefore$ the roots of the polynomial are $x = \tan\theta$, where $\tan 4\theta = -1$.



$$\tan 4\theta = -1$$

$$4\theta = \pi - \tfrac{\pi}{4}, 2\pi - \tfrac{\pi}{4}, 3\pi - \tfrac{\pi}{4}, 4\pi - \tfrac{\pi}{4}$$

$$= \tfrac{3\pi}{4}, \tfrac{7\pi}{4}, \tfrac{11\pi}{4}, \tfrac{15\pi}{4}$$

$$\theta = \tfrac{3\pi}{16}, \tfrac{7\pi}{16}, \tfrac{11\pi}{16}, \tfrac{15\pi}{16}$$

$\therefore$ roots are $\tan\tfrac{3\pi}{16}, \tan\tfrac{7\pi}{16}, \tan\tfrac{11\pi}{16}, \tan\tfrac{15\pi}{16}$. ✓

(Accept θ in any domain such that all $\tan\theta$ are unique)

Question 14

(a) (i) $\ddot{x} = -G + kv^2$

When $v = -40$, $\ddot{x} = 0$:

$$0 = -G + k \times (-40)^2$$

$$\frac{G}{k} = 40^2$$

$$\sqrt{\frac{G}{k}} = 40$$ ✓

(ii) (α) $\ddot{x} = -G - kv^2$

$$\frac{dv}{dt} = -(G + kv^2)$$

$$\int \frac{dv}{G + kv^2} = -\int dt$$

$$\frac{1}{k} \int \frac{dv}{\frac{G}{k} + v^2} = -\int dt$$

$$\int \frac{dv}{40^2 + v^2} = -k \int dt$$

$$\frac{1}{40} \tan^{-1}\left(\frac{v}{40}\right) = -kt + c$$ ✓

When $t = 0$, $v = 40$: $c = \frac{1}{40} \times \tan^{-1}(1)$

$$= \frac{\pi}{160}$$

$$\tan^{-1}\left(\frac{v}{40}\right) = 40\left(\frac{\pi}{160} - kt\right)$$

$$\frac{v}{40} = \tan\left(\frac{\pi}{4} - 40kt\right)$$

$$v = 40\tan\left(\frac{\pi}{4} - 40kt\right)$$ ✓

(β) When $t = 4$, $v = 0$

$$\therefore \tan\left(\frac{\pi}{4} - 160k\right) = 0$$

$$\frac{\pi}{4} - 160k = 0$$

$$k = \frac{\pi}{640}$$ ✓

$$\frac{G}{k} = 40^2$$

$$G = 40^2 \times k = 40^2 \times \frac{\pi}{640} = \frac{5\pi}{2} \text{ ms}^{-2}$$ ✓

(b) (i) $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$ (cos even, sin odd)

$= 2\cos\theta$ ✓

(ii) Since $|z_3| = \frac{1}{\sqrt{3}}$,

$$\left|\frac{1}{z_1 + z_2}\right| = \frac{1}{\sqrt{3}}$$

$\therefore |z_1 + z_2| = \sqrt{3}$ ✓

$|e^{i\alpha} + e^{i\beta}| = \sqrt{3}$

$$\therefore \left|e^{i(\alpha+\beta)/2}\left(e^{i(\alpha-\beta)/2} + e^{-i(\alpha-\beta)/2}\right)\right| = \sqrt{3}$$

$$\left|e^{i(\alpha+\beta)/2}\right| \times \left|e^{i(\alpha-\beta)/2} + e^{i(\alpha-\beta)/2}\right| = \sqrt{3}$$

$$1 \times \left|2\cos\left(\frac{\alpha-\beta}{2}\right)\right| = \sqrt{3}$$ ✓ (from (i))

$$\left|\cos\left(\frac{\alpha-\beta}{2}\right)\right| = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\alpha-\beta}{2} = \pm\frac{\sqrt{3}}{2}$$

Since $0 \le \alpha, \beta \le \pi$, $-\frac{\pi}{2} \le \frac{\alpha-\beta}{2} \le \frac{\pi}{2}$

$\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{\sqrt{3}}{2}$ $\quad\quad$ $\cos\left(\frac{\alpha-\beta}{2}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow \frac{\alpha-\beta}{2} = \frac{\pi}{6}$ $\quad\quad$ $\Rightarrow \frac{\alpha-\beta}{2} = -\frac{\pi}{6}$

$\alpha - \beta = \frac{\pi}{3}$ $\quad\quad$ $\alpha - \beta = -\frac{\pi}{3}$ ✓ (both values)

(c) (i) Base case: Prove true for $n=1$.

$x^n + 1 = x + 1$

$= 1(x+1)$

$\therefore$ true for $n=1$

Suppose statement is true for some $n=k$.

i.e. $x^k + 1 = M \times (x+1)$, where $M \in \mathbb{Z}$. $(*)$

Prove true for $n=k+2$:

RTP: $x^{k+2} + 1$ is divisible by $x+1$.

$x^{k+2} + 1 = x^2 \times x^k + 1$

$= x^2(M(x+1) - 1) + 1$ from $(*)$ ✓

$= x^2 M(x+1) - x^2 + 1$

$= Mx^2(x+1) - (x^2 - 1)$

$= Mx^2(x+1) - (x+1)(x-1)$

$= (x+1)(Mx^2 - (x-1))$ ✓

$\therefore$ true for $n=k+2$.

$\therefore$ statement is true for all positive odd integers by mathematical induction.

(ii) $n = m \times 2^k$

Contrapositive: If m does not have an odd factor larger than 1, then n is a power of 2. ✓

If m doesn't have an odd factor >1, then m is a power of 2. $\therefore$ Let $m = 2^j$, $j \in \mathbb{Z}^+$.

$\therefore n = 2^j \times 2^k$

$= 2^{j+k}$, ✓ which is a power of 2.

$\therefore$ By the contrapositive, if n is not a power of 2, then m has an odd factor larger than 1.

(iii) Again, consider the contrapositive:

"If n is not a power of 2, then 2^n+1 is not prime".

From part (ii), if n is not a power of 2, then $n = m \times 2^k$, where m has an odd factor > 1.

Let $n = a \times b$, where a is odd and $a > 1$, and $b > 1$.

Then $2^n + 1 = 2^{ab} + 1$

$= (2^b)^a + 1$

From part (i), this is divisible by $2^b + 1$ since a is odd. Also, since $b > 1$, $2^b + 1 > 1$

$\therefore (2^b)^a + 1 = 2^n + 1$ is not prime.

By contraposition, if $2^n + 1$ is prime, then n is a power of 2.

Question 15

(a) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

(i) $\int_0^1 (1-\sqrt{x})^n \, dx = \int_0^1 1 \times (1-\sqrt{x})^n \, dx$

$v' = 1 \quad u = (1-\sqrt{x})^n$

$v = x \quad u' = n(1-\sqrt{x})^{n-1} \times -\frac{1}{2}x^{-\frac{1}{2}}$

$= \left[x(1-\sqrt{x})^n\right]_0^1 + \lim_{m \to 0} \frac{n}{2} \int_m^1 x(1-\sqrt{x})^{n-1} \times x^{-\frac{1}{2}} \, dx$ ✓

$= \frac{n}{2} \int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1} \, dx$ ✓

$= \frac{n}{2} \int_0^1 -((1-\sqrt{x})-1)(1-\sqrt{x})^{n-1} \, dx$

$= -\frac{n}{2} \int_0^1 \left((1-\sqrt{x})^n - (1-\sqrt{x})^{n-1}\right) dx$

$I_n = -\frac{n}{2}(I_n - I_{n-1})$

$I_n\left(1+\frac{n}{2}\right) = \frac{n}{2} I_{n-1}$

$I_n\left(\frac{n+2}{2}\right) = \frac{n}{2} I_{n-1}$

$I_n = \frac{n}{n+2} I_{n-1}$ ✓

(ii) $I_n = \frac{n}{n+2} I_{n-1}$

$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times I_{n-2}$

$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times I_{n-3}$

$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \frac{n-3}{n-1} \times \ldots \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times I_0$

$= \frac{1}{n+2} \times \frac{1}{n+1} \times 2 \times 1 \quad (\text{since } I_0 = 1)$

$= \frac{2}{(n+2)(n+1)}$ ✓ (must address I_0)

(iii) Let $\frac{2}{(n+1)(n+2)} \equiv \frac{A}{n+1} + \frac{B}{n+2}$

$2 \equiv A(n+2) + B(n+1)$

$n=-1$: $A = 2$

$n=-2$: $B = -2$

$\therefore \frac{2}{(n+1)(n+2)} = \frac{2}{n+1} - \frac{2}{n+2}$ ✓

$I_0 + I_1 + I_2 + I_3 + \dots + I_{n-1} + I_n$

$= \left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \dots + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{2}{n+1} - \frac{2}{n+2}\right)$

$= 2 - \frac{2}{n+2}$

< 2 since $n > 0$ ✓

(b) $\int \frac{p-x}{\sqrt{q^2+(p-x)^2}}\,dx$ Let $u = (p-x)^2$

$du = -2(p-x)\,dx$

$= -\frac{1}{2}\int \frac{-2(p-x)\,dx}{\sqrt{q^2+(p-x)^2}}$

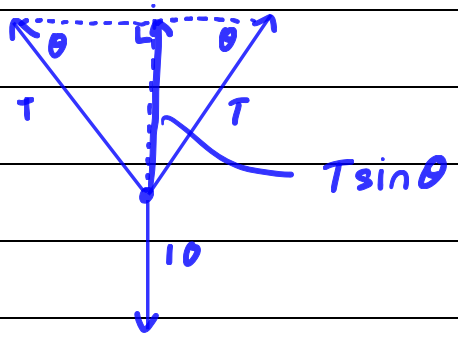
$= -\frac{1}{2}\int (q^2+u)^{-\frac{1}{2}}\,du$

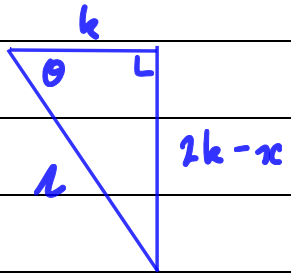
$= -\frac{1}{2} \times \frac{1}{\frac{1}{2}} (q^2+u)^{\frac{1}{2}} + c$

$= -\sqrt{q^2+u} + c$

$= -\sqrt{q^2+(p-x)^2} + c$ ✓

(c) (i)





$$L = \sqrt{k^2 + (2k-x)^2}$$

$$\sin\theta = \frac{2k-x}{\sqrt{k^2+(2k-x)^2}}$$ ✓

Since mass = 1kg,

$$\ddot{x} = 2T\sin\theta - 10$$

$$= \frac{2T(2k-x)}{\sqrt{k^2+(2k-x)^2}} - 10$$ ✓

(ii) $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{2T(2k-x)}{\sqrt{k^2+(2k-x)^2}} - 10$$

$$\frac{1}{2}v^2 = -2T\sqrt{k^2+(2k-x)^2} - 10x + c$$ ✓ from part (a)

$$v^2 = -4T\sqrt{k^2+(2k-x)^2} - 20x + c_2$$

When $x = 0$, $v = 0$: $c_2 = 4T\sqrt{5k^2}$

$$= 4Tk\sqrt{5}$$

$$\therefore v^2 = 4T\left(k\sqrt{5} - \sqrt{k^2+(2k-x)^2}\right) - 20x$$

i.e. $$(\dot{x})^2 = 4T\left(k\sqrt{5} - \sqrt{k^2+(2k-x)^2}\right) - 20x$$ ✓

(iii) If object touches the ceiling, $\dot{x} > 0$ when $x = 2k$.

i.e. $$4T\left(k\sqrt{5} - \sqrt{k^2}\right) - 20 \times 2k > 0$$

$$T > \frac{40k}{4k(\sqrt{5}-1)} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}$$

$$T > \frac{10(\sqrt{5}+1)}{4}$$

$$T > 5\varphi, \text{ where } \varphi = \frac{\sqrt{5}+1}{2}$$ ✓

(iv) Maximum velocity reached when $\ddot{x} = 0$.

Using $T = 5\varphi$:

$$\frac{10\varphi(2k-x)}{\sqrt{k^2+(2k-x)^2}} - 10 = 0$$ ✓

$$\varphi(2k-x) = \sqrt{k^2+(2k-x)^2}$$

$$\varphi^2(2k-x)^2 = k^2+(2k-x)^2$$

$$(2k-x)^2(\varphi^2-1) = k^2$$

$$(2k-x)^2 = \frac{k^2}{\varphi^2-1}$$

$$= \frac{k^2}{\varphi}$$ ✓

$$2k-x = \frac{k}{\sqrt{\varphi}} \quad \text{since } 0 \le x \le 2k \text{ and } k > 0$$

$$x = \frac{2k\sqrt{\varphi}-k}{\sqrt{\varphi}}$$

∴ Fraction of way between floor and ceiling:

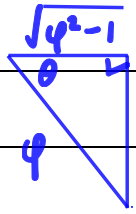
$$p = \frac{\frac{k(2\sqrt{\varphi}-1)}{\sqrt{\varphi}}}{2k}$$

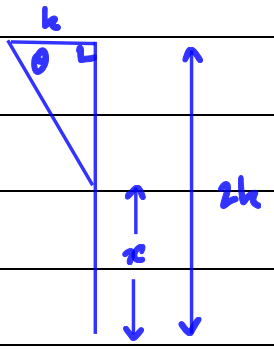
$$= \frac{2\sqrt{\varphi}-1}{2\sqrt{\varphi}}$$ ✓

Alternatively: $\ddot{x} = 2T\sin\theta - 10$

$\ddot{x} = 0$, $T = 5\varphi$: $10\varphi\sin\theta - 10 = 0$

$$\sin\theta = \frac{1}{\varphi}$$





$$p = 1 - \frac{2k-x}{2k}$$

$$= 1 - \frac{1}{2} \times \frac{2k-x}{k}$$

$$= 1 - \frac{1}{2} \times \tan\theta$$

$$= 1 - \frac{1}{2} \times \frac{1}{\sqrt{\varphi^2-1}}$$

$$= \frac{2\sqrt{\varphi}-1}{2\sqrt{\varphi}}$$

Question 16

(a) (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(\sin x)\,dx$ $\quad v' = 1 \quad u = f(\sin x)$ $\quad v = x \quad u' = f'(\sin x)\cos x$

$= \left[x f(\sin x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos x\, f'(\sin x)\,dx$ ✓

$= \frac{\pi}{2} f(\sin\frac{\pi}{2}) - \frac{\pi}{6} f(\sin\frac{\pi}{6}) - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos x\, f'(\sin x)\,dx$

$= \frac{\pi}{6}\left(3f(1) - f(\tfrac{1}{2}) \right) - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos x\, f'(\sin x)\,dx$ ✓

(ii) From (i): $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(f(\sin x) + x\cos x\, f'(\sin x) \right) dx = \frac{\pi}{6}\left(3f(1) - f(\tfrac{1}{2})\right)$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2\sin x - x\cos x}{\sqrt{\sin x}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sqrt{\sin x} - \frac{x\cos x}{\sqrt{\sin x}} \right) dx$

Let $f(x) = 2\sqrt{x}$ ✓

$f'(x) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

So $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sqrt{\sin x} - \frac{x\cos x}{\sqrt{\sin x}} \right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(f(\sin x) + x\cos x\, f'(\sin x) \right) dx$

(where $f(x) = 2\sqrt{x}$)

$= \frac{\pi}{6}\left(3f(1) - f(\tfrac{1}{2}) \right)$

$= \frac{\pi}{6}\left(3 \times 2\sqrt{1} - 2\sqrt{\tfrac{1}{2}} \right)$

$= \frac{\pi}{6}\left(6 - \sqrt{2} \right)$ ✓

(b) (i) Consider the numbers:

$$\underbrace{a_1, a_1, a_1, \dots, a_1}_{k_1}, \underbrace{a_2, a_2, \dots, a_2}_{k_2}, \dots, \underbrace{a_n, a_n, \dots, a_n}_{k_n}$$ ✓

Applying AM-GM:

$$\frac{a_1 + a_1 + \dots + a_2 + a_2 + \dots + a_n + a_n}{k_1 + k_2 + \dots + k_n} \geq \left(a_1 \times a_1 \times \dots \times a_2 \times a_2 \times \dots \times a_n \times a_n \times \dots\right)^{\frac{1}{k_1 + k_2 + \dots + k_n}}$$

$$\therefore \frac{a_1 k_1 + a_2 k_2 + \dots + a_n k_n}{k_1 + k_2 + \dots + k_n} \geq \left(a_1^{k_1} a_2^{k_2} \dots a_n^{k_n}\right)^{\frac{1}{k_1 + k_2 + \dots + k_n}}$$

(ii) Let $a_1 = a$, $k_1 = a$, $a_2 = b$, $k_2 = b$, $a_3 = c$, $k_3 = c$:

$$\frac{a^2 + b^2 + c^2}{a + b + c} \geq (a^a b^b c^c)^{\frac{1}{a+b+c}}$$

Since $a+b+c = 1$: $a^2 + b^2 + c^2 \geq a^a b^b c^c$ (1) ✓

Let $a_1 = a$, $k_1 = b$, $a_2 = b$, $k_2 = c$, $a_3 = c$, $k_3 = a$:

$$\frac{ab + bc + ac}{a + b + c} \geq (a^b b^c c^a)^{\frac{1}{a+b+c}}$$

Since $a+b+c = 1$: $ab + bc + ac \geq a^b b^c c^a$ (2) ✓

Let $a_1 = a$, $k_1 = c$, $a_2 = b$, $k_2 = a$, $a_3 = c$, $k_3 = b$:

$$\frac{ac + ab + bc}{a + b + c} \geq (a^c b^a c^b)^{\frac{1}{a+b+c}}$$

Since $a+b+c = 1$: $ac + ab + bc \geq a^c b^a c^b$ (3)

(1) + (2) + (3): $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq a^2 + b^2 + c^2 + 2(ab + ac + bc)$

$= (a+b+c)^2$ ✓

$= 1$ since $a+b+c = 1$

(c) (i) Solving simultaneously: $\begin{pmatrix} p\cos p \\ p\sin p \\ \frac{1}{k}p^2 \end{pmatrix} = \begin{pmatrix} q\cos q \\ -q\sin q \\ \frac{1}{k}q^2 \end{pmatrix}$

$\underset{\sim}{i}$ and $\underset{\sim}{k}$ coordinates equal when $p=q$, but $\underset{\sim}{j}$ coordinates will only be equal when $p=q$ and coordinates are 0.

i.e. $p\sin p = 0$, so R_1 is denoted by $p=\pi$,

R_2 by 2π, etc.

$\therefore$ R_n has parameter $n\pi$. ✓

(ii)(α) $R_n = \begin{pmatrix} n\pi\cos(n\pi) \\ n\pi\sin(n\pi) \\ \frac{1}{k}(n\pi)^2 \end{pmatrix} = \begin{pmatrix} n\pi\cos(n\pi) \\ 0 \\ \frac{1}{k}(n\pi)^2 \end{pmatrix}$

$R_{n+1} = \begin{pmatrix} (n+1)\pi\cos((n+1)\pi) \\ 0 \\ \frac{1}{k}((n+1)\pi)^2 \end{pmatrix} = \begin{pmatrix} -(n+1)\pi\cos(n\pi) \\ 0 \\ \frac{1}{k}((n+1)\pi)^2 \end{pmatrix}$

$\overrightarrow{R_nR_{n+1}} = \begin{pmatrix} -(n+1)\pi\cos(n\pi) - n\pi\cos(n\pi) \\ 0 \\ \frac{1}{k}((n+1)\pi)^2 - \frac{1}{k}(n\pi)^2 \end{pmatrix}$

$= \begin{pmatrix} -(2n+1)\pi\cos(n\pi) \\ 0 \\ \frac{1}{k}\pi^2(2n+1) \end{pmatrix}$ ✓

$\therefore \overrightarrow{R_{n+1}R_{n+2}} = \begin{pmatrix} -(2(n+1)+1)\pi\cos((n+1)\pi) \\ 0 \\ \frac{1}{k}\pi^2(2(n+1)+1) \end{pmatrix}$

$= \begin{pmatrix} (2n+3)\pi\cos(n\pi) \\ 0 \\ \frac{1}{k}\pi^2(2n+3) \end{pmatrix}$

$\overrightarrow{R_nR_{n+1}} \cdot \overrightarrow{R_{n+1}R_{n+2}} = -(2n+1)(2n+3)\pi^2\cos^2(n\pi) + \frac{\pi^4}{k^2}(2n+1)(2n+3)$ ✓

Since $\overrightarrow{R_nR_{n+1}} \perp \overrightarrow{R_{n+1}R_{n+2}}$:

$-(2n+1)(2n+3)\pi^2\cos^2(n\pi) + \frac{\pi^4}{k^2}(2n+1)(2n+3) = 0$

As $\cos^2(n\pi) = 1$: $\quad \frac{\pi^2}{k^2} = 1$

$k^2 = \pi^2$

$k = \pi$ ✓ (since $k > 0$)

* Note: Award full marks if a pupil uses $\overrightarrow{R_1R_2} \cdot \overrightarrow{R_2R_3} = 0$ (i.e. picking a specific n)

(ii)(β) Let the direction vector of l_1 be $\underset{\sim}{d_1}$, and l_2 be $\underset{\sim}{d_2}$.

At R_n:

$$\underset{\sim}{d_1} = \begin{pmatrix} -n\pi\sin(n\pi) + \cos(n\pi) \\ n\pi\cos(n\pi) + \sin(n\pi) \\ \frac{2}{\pi} \times n\pi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(n\pi) \\ n\pi\cos(n\pi) \\ 2n \end{pmatrix}$$

$$\underset{\sim}{d_2} = \begin{pmatrix} -n\pi\sin(n\pi) + \cos(n\pi) \\ -n\pi\cos(n\pi) - \sin(n\pi) \\ \frac{2}{\pi} \times n\pi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(n\pi) \\ -n\pi\cos(n\pi) \\ 2n \end{pmatrix}$$

✓ simplified form of direction vectors.

$$\cos\theta_n = \frac{\underset{\sim}{d_1} \cdot \underset{\sim}{d_2}}{|\underset{\sim}{d_1}| \times |\underset{\sim}{d_2}|}$$

$$= \frac{\cos^2(n\pi) - n^2\pi^2\cos^2(n\pi) + 4n^2}{\left(\sqrt{\cos^2(n\pi) + n^2\pi^2\cos^2(n\pi) + 4n^2}\right)^2}$$

$$= \frac{1 - n^2\pi^2 + 4n^2}{1 + n^2\pi^2 + 4n^2}$$ ✓

Note that $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\cos\theta_n = \frac{1+4n^2-n^2\pi^2}{1+4n^2+n^2\pi^2}$$

$$= \frac{1-\frac{n^2\pi^2}{1+4n^2}}{1+\frac{n^2\pi^2}{1+4n^2}}$$

$$\therefore \tan^2\frac{\theta_n}{2} = \frac{n^2\pi^2}{1+4n^2}$$

$$\tan\frac{\theta_n}{2} = \frac{n\pi}{\sqrt{1+4n^2}} \quad \text{since } \theta_n \text{ is obtuse so } \tan\frac{\theta_n}{2} > 0.$$

$$= \frac{n\pi}{n\sqrt{\frac{1}{n^2}+4}}$$

$$= \frac{\pi}{\sqrt{\frac{1}{n^2}+4}}$$

$$\therefore \lim_{n\to\infty} \tan\frac{\theta_n}{2} = \frac{\pi}{\sqrt{4}}$$

$$= \frac{\pi}{2} \quad \checkmark$$