Sydney Technical High School

Name: .....

Maths Class: .....

Year 12

Mathematics Extension 2

HSC Course

Trial

August, 2026

Time allowed: 180 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Section I Multiple Choice
Questions 1-10
10 Marks

Section II Questions 11-16
90 Marks

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. What is the scalar product of $19\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}$ and $3\underset{\sim}{i} + 8\underset{\sim}{j} - 7\underset{\sim}{k}$

A. 66

B. 67

C. 76

D. 77

2. $\overrightarrow{OA} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, which of the following vectors is parallel to $\overrightarrow{AB}$?

A. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

3. With a suitable substitution $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x \ dx$ can be expressed as:

A. $\int_1^2 (u^5 + u^7)\, du$

B. $\int_0^{\sqrt{3}} (u^5 + u^7)\, du$

C. $\int_0^{\sqrt{3}} u^3(u^2 - 1)^2\, du$

D. $\int_1^2 (u^4 + u^6)\, du$

4. For any two complex numbers z_1 and z_2 which of the following inequalities is always true?

A. $|z_1 - z_2| \geq |z_1| + |z_2|$

B. $|z_1 + z_2| \leq |z_1| - |z_2|$

C. $|z_1 - z_2| \leq \big||z_1| - |z_2|\big|$

D. $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$

5. $f(x)$ is continuous, differentiable, and integrable over its domain.

$$\int_0^{\frac{\pi}{2}} f(x)\cos x \; dx = 2, \quad f(0) = -1, \text{ and } f\left(\frac{\pi}{2}\right) = 3$$

Evaluate $I = \int_0^{\frac{\pi}{2}} f'(x)\sin x \, dx$

A. $I = 1$

B. $I = 2$

C. $I = 5$

D. $I = 6$

6. Which of the following functions does NOT describe simple harmonic motion

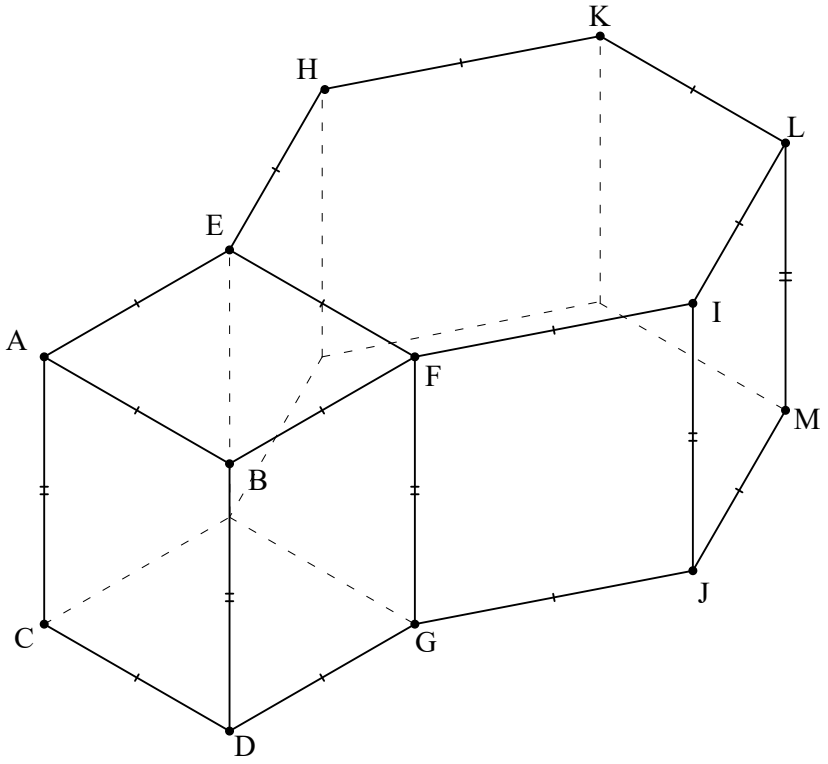
A. $4\sin^2 t + 3\cos 2t$

B. $3\sin\left(3t + \frac{\pi}{6}\right) - 2\cos\left(3t - \frac{\pi}{3}\right)$

C. $5\sin 4t + 12\cos 4t - 3$

D. $3\sin 5t - 2\cos 3t$

7.



A 3D shape is formed by placing two right prisms side by side, as shown in the diagram. One prism has a square base $ABFE$ and the other has a regular hexagonal base $EFILKH$. Both prisms have the same height, and all base edges have the same length.

Which of the following represents $\overrightarrow{DK}$

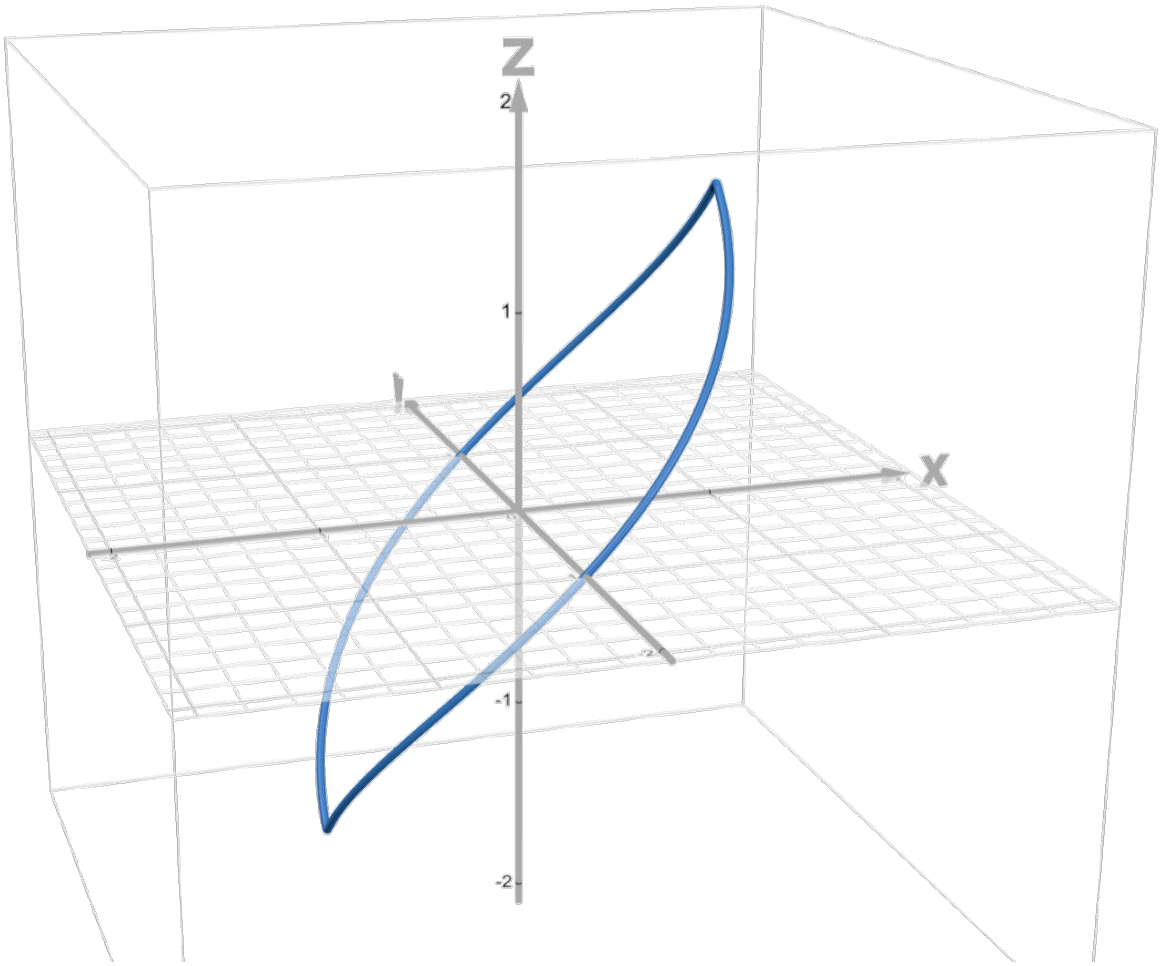
A. $\overrightarrow{DK} = \overrightarrow{AB} - \overrightarrow{AC} + \overrightarrow{AE} - \overrightarrow{HE} + \overrightarrow{HK}$

B. $\overrightarrow{DK} = \overrightarrow{DG} + \overrightarrow{GJ} + \overrightarrow{IL} - \overrightarrow{KL}$

C. $\overrightarrow{DK} = \overrightarrow{AE} + \overrightarrow{HK} - \overrightarrow{AB} - \overrightarrow{FG} + \overrightarrow{IL}$

D. $\overrightarrow{DK} = \overrightarrow{AE} + \overrightarrow{HK} + \overrightarrow{AB} - \overrightarrow{AC} + \overrightarrow{IL}$

8.



Which of the following best represents the parametric equation of the $3D$ curve shown above?

A. $(\cos \pi t\ ,\ \sin \pi t\ , \arcsin(\sin \pi t)), -2\pi \leq t \leq 2\pi$

B. $(\sin \pi t\ ,\ \cos \pi t\ , \arcsin(\sin \pi t)), -2\pi \leq t \leq 2\pi$

C. $(\sin \pi t\ ,\ \cos \pi t\ , \arccos(\cos \pi t)), -2\pi \leq t \leq 2\pi$

D. $(\cos \pi t\ ,\ \arccos(\cos \pi t)\ , \sin \pi t), -2\pi \leq t \leq 2\pi$

9. The intersection of four regions on the Argand diagram forms a kite with diagonals running along the real and imaginary axes. The regions are defined by:

$$\arg(z - i) \leq -\frac{\pi}{3}$$

$$\arg(z + i) \leq \frac{\pi}{3}$$

$$\arg\left(z + \sqrt{3}\right) \leq \frac{\pi}{6}$$

$$\arg\left(z + \sqrt{3}\right) \geq -\frac{\pi}{6}$$

What is the exact area of the kite?

A. $\frac{2}{\sqrt{3}}$

B. $\frac{4}{\sqrt{3}}$

C. $\frac{8}{\sqrt{3}}$

D. $2\sqrt{3}$

10. $f(x)$ is a continuous and integrable function over the real numbers.

Given that $f(2x) = 3f(x)$ and $\int_0^1 f(x)\,dx = 2$

find $\int_1^2 f(x)\,dx$

A. 4

B. 6

C. 10

D. 12

Please turn over for Section II

Section II

Attempt Questions 11-16 (90 marks)
Allow about 2 hours and 45 minutes for this section

Answer each question on a new page in the answer booklet

For questions in Section II, your responses **must include** relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new page **Marks**

a) Find $\int 2\cos 5x \cos 4x \ dx$ **2**

b) **i)** Write the number $1-\sqrt{3}i$ in exponential form **2**

ii) Hence, or otherwise write $\left(1-\sqrt{3}i\right)^6$ in exact cartesian form **2**

c) Find the values of α, and β such that the points $(-3, \alpha, \beta)$, $(5, 4, 2)$, and $(3, 12, 2)$ are collinear **2**

d) Solve the quadratic equation $z^2 - 6z + 13 = 0$ over the complex numbers giving your answer in Cartesian form **2**

e) O and A are two points such that $\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$ and $\overrightarrow{OB} = 5\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$

i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to the nearest degree **3**

ii) Find the exact area of triangle OAB **2**

End of Question 11

Question 12 (15 marks) Start a new page **Marks**

a) Evaluate $\int_1^e \frac{(\ln x)^3}{x}\, dx$ **2**

b) Solve $z = \sqrt[3]{i}$, giving your answers in cartesian form **3**

c) Shade the region in the complex plane defined by **3**

$$1 < \text{Im}(z) \leq 4 \quad \text{and} \quad \frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$$

d) A particle is moving in simple harmonic motion described by $\ddot{x} = -9(x-2)$ **3**

When the particle passes through the point $x = -1$ m its speed is $6\sqrt{3}$ ms^{-1}

Find the distance the particle travels during a full period of its motion

e) The points $A = (-1, 0, 0)$, B $(0, 1, 0)$, and C $(0, 0, 1)$ are 3 of the vertices of a regular unit octahedron centred on the origin

i) Find $\text{proj}_{\overrightarrow{BC}} \overrightarrow{AB}$ **2**

ii) Hence or otherwise find the shortest distance from A to the line BC **2**

End of Question 12 please turn over

Question 13 (15 marks) Start a new page **Marks**

a) Evaluate $\sum_{n=1}^{2026} i^n$ **2**

b) i) Show that $x + \frac{1}{x} \geq 2$ for all positive real numbers x **1**

ii) Prove that for four positive real numbers a, b, c, d , $a \neq b \neq c \neq d$ **2**

$$\left(\frac{a}{b} - \frac{c}{d}\right)\left(\frac{b}{a} - \frac{d}{c}\right) \leq 0$$

c) Use the substitution $u^2 = 5 - x$, $u > 0$ to show that **4**

$$\int_1^4 \frac{3}{x + 4\sqrt{5 - x}}\, dx = \ln\left(\frac{2048}{729}\right)$$

d) Consider the complex number $z = \cos\theta - i\sin\theta$

i) Use De Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2\cos n\theta\,, \quad n \in \mathbb{Z}$ **2**

ii) Hence or otherwise express $\left(z + \frac{1}{z}\right)^5$ in the form: **2**

$$A\cos(5\theta) + B\cos(3\theta) + C\cos(\theta) \quad \text{where } A, B, C \in \mathbb{R}$$

iii) Hence or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \cos^5\theta \; d\theta$ **2**

End of Question

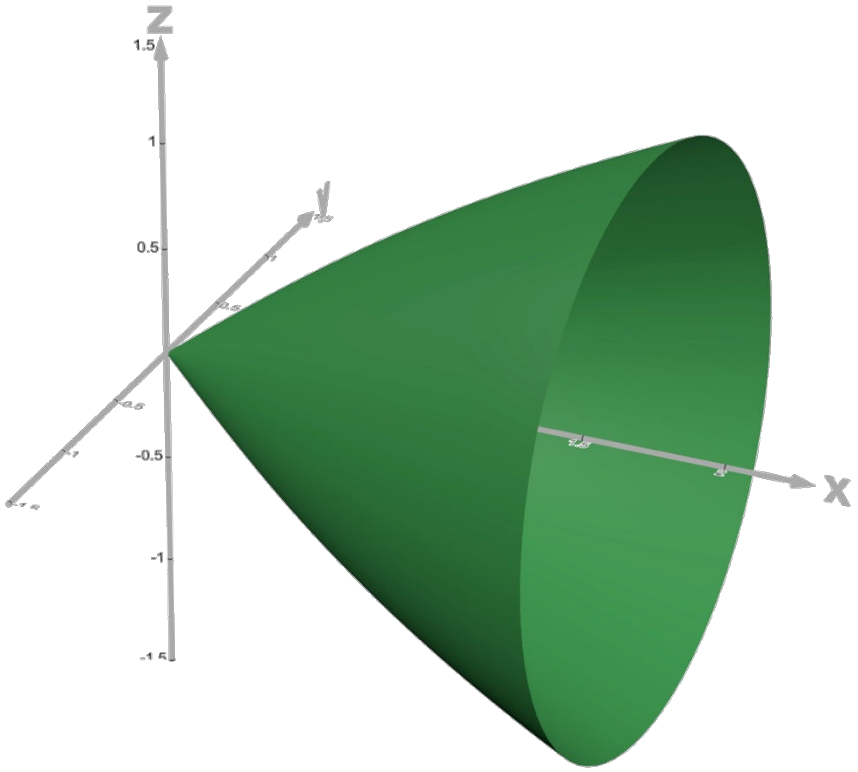
Question 14 (15 marks) Start a new page **Marks**

a) A particle initially at the origin moves in a straight line with an initial velocity of 1 m/s such that its acceleration a m/s^2 is given by $a = v\sqrt{4 - v^2}$

i) Find its exact velocity when its displacement $x = \frac{\pi}{6}$ **3**

ii) With relevant mathematical reasoning, describe the eventual motion of the particle **1**

b)



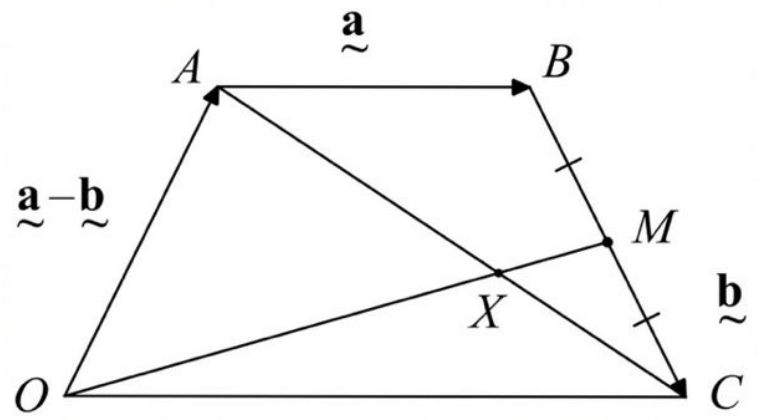
Find the exact volume of the solid formed when $y = \sqrt{2x \arctan x}$ is rotated about the $x-$ axis, between $x = 0$ and $x = \sqrt{3}$ **3**

Question 14 continues over the page

Question 14 continues ... **Marks**

c) Prove by contradiction that $4x^2 - y^2 = 1$ does not have any integer solutions **3**

d) $OABC$ is a quadrilateral



$\overrightarrow{AB} = \underset{\sim}{\boldsymbol{a}}$, $\overrightarrow{BC} = \underset{\sim}{\boldsymbol{b}}$, and $\overrightarrow{OA} = \underset{\sim}{\boldsymbol{a}} - \underset{\sim}{\boldsymbol{b}}$

M is the midpoint of BC

X is the point on OM such that $OX : XM = n : 1$

X also lies on the line AC

i) Express $\overrightarrow{OC}$ in terms of $\underset{\sim}{\boldsymbol{a}}$ and/or $\underset{\sim}{\boldsymbol{b}}$ in its simplest form **1**

ii) Express $\overrightarrow{OX}$ in terms of $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ and hence show that **2**

$$\overrightarrow{AX} = \frac{n-1}{n+1}\underset{\sim}{\boldsymbol{a}} + \frac{n+2}{2(n+1)}\underset{\sim}{\boldsymbol{b}}$$

iii) Find the value of n **2**

End of Question

Question 15 (15 marks) Start a new page **Marks**

a) What is the contrapositive of the following implication **1**

$$\neg(A \Longrightarrow C) \Longrightarrow \big(\neg(A \Longrightarrow B)\ OR\ \neg(B \Longrightarrow C)\big)$$

b) Let $I_n = \int_0^1 x(1-x)^n\,dx$ where n is positive integer

i) Show that $I_n = \dfrac{1}{(n+1)(n+2)}$ **3**

ii) Hence by considering the partial fraction decomposition of I_n, find a closed form algebraic expression for S_n , the sum of the first n terms of I_n **2**

$$S_n = \sum_{k=1}^{n} I_k$$

c) Use mathematical induction to prove that, for all positive integers n **4**

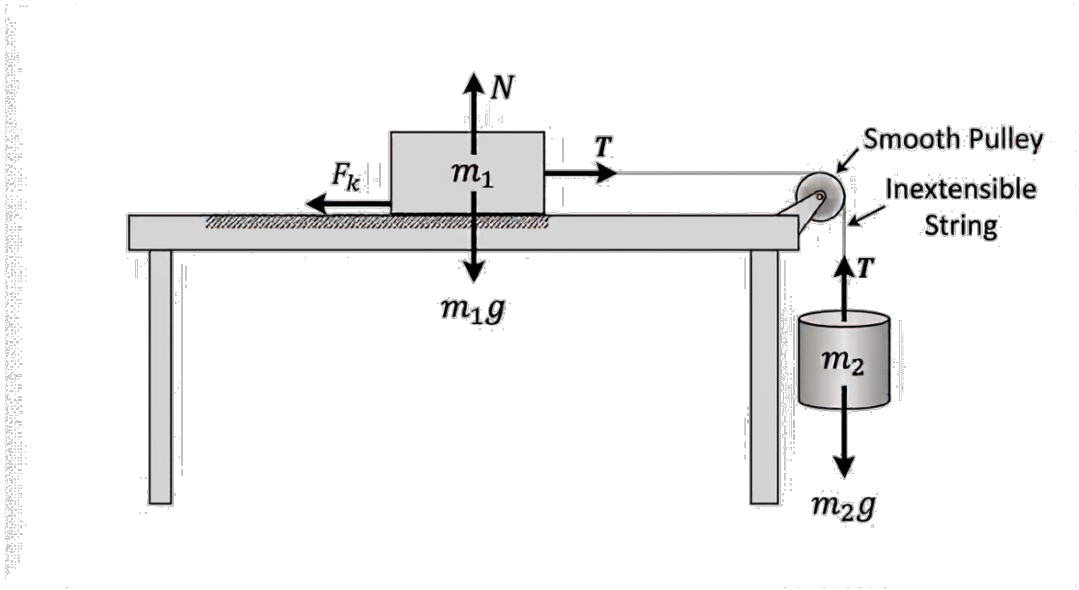
$\tan\theta + 2\tan 2\theta + \ \ldots\ + 2^{n-1}\tan(2^{n-1}\theta) \ = \ \cot\theta - 2^n\cot(2^n\theta)$

Question 15 continues over the page

d) Use the fundamental theorem of calculus to show that $\frac{d}{dx}\left[\int_1^{x^2} \ln t \, dt\right] = 4x \ln x$ **2**

e) A light, inextensible string passes over a smooth, fixed pulley. **3**

One end is attached to a mass $m_1 = 8$ kg on a rough horizontal table, where a kinetic friction force $F_k = u_k N$ newtons acts in the opposite direction to that of the motion, where u_k is the coefficient of kinetic friction and N is the normal force acting on m_1. The other end of the string hangs vertically and is attached to a mass $m_2 = 6$ kg



The coefficient of kinetic friction $u_k = \frac{1}{4}$

The system is released from rest

In terms of gravity g, what is the tension T in the string ?

End of Question 15

Question 16 (15 marks) Start a new page **Marks**

a) i) Prove that the product of two consecutive positive even integers is divisible by 8 **2**

ii) Hence prove that for all prime numbers $P > 3$, $P^2 - 1$ is divisible by 24 **3**

b) An exploratory lander on the distant exoplanet Arrakis uses a high-powered magnetic launcher to fire a sampling probe of mass m kg vertically upwards. The launcher is positioned at $x = 1$, where x represents the displacement from the surface in meters.

Driven by magnetic propulsion from the launcher, the probe moves through a dense plasma atmosphere as it travels upwards under a gravity of g m/s^2

The combined net non-gravitational force F_r acting on the metal probe is modelled by

$$F_r = \frac{mg}{x^3}(1 + v^2) - mgv^2$$

Where v is the speed of the probe

The lander's booster launches the probe with an initial upward speed of U m/s

i) Show that the equation of motion while the probe is moving upwards can be written as: **1**

$$\frac{v}{1 + v^2}\frac{dv}{dx} = -g\left(1 - \frac{1}{x^3}\right)$$

ii) Hence, show that the greatest height of the probe $x = H$, satisfies: **3**

$$\ln\,(1 + U^2) = 2g(H - 1) - g\left(\frac{H^2 - 1}{H^2}\right)$$

Question 16 continues over the page

c) A and B are two distinct points in three-dimensional space such that

$$\overrightarrow{OA} = \begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix}$$

i) Let S_1 be the set of all points P such that $\overrightarrow{AP} \,.\, \overrightarrow{BP} = 0$. **2**

Find the vector equation of S_1

ii) let M be the midpoint of the line AB and S_2 be the set of all points N such that **1**

$|\overrightarrow{AN}| = |\overrightarrow{MN}|$

The intersection of S_1 and S_2 is the circle S. Find the radius of S

d) **i)** Show that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ **1**

ii) Hence or otherwise find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx$ **2**

End of Exam

2026 EX2 Trial solutions

1	2	3	4	5	6	7	8	9	10
B	A	B	D	A	D	C	B	B	C

1) $19\times3 - 4\times8 + 6\times7 = 67$ $\therefore$ (B)

2) $AB = \begin{bmatrix} 1 & -2 \\ -2 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\therefore$ (A)

3) $I = \int_0^{\pi/3} \tan^5 x \sec^4 x \, dx$

let $u = \tan x$

$du = \sec^2 x \, dx$

$dx = \frac{du}{\sec^2 x}$

$x = 0 \quad u = 0$

$x = \pi/3 \quad u = \sqrt{3}$

$= \int_{u=0}^{\sqrt{3}} u^5 \cdot (1+\tan^2 x) \frac{\sec^2 x}{\sec^2 x} du$

$= \int_0^{\sqrt{3}} u^5 (1+u^2) \, du$ $\therefore$ (B)

4) A false as $|z_1 - z_2| < |z_1| + |z_2|$ e.g. if signs same i.e. $z_1 = 1$ $z_2 = 1$

B false again if signs same.

e.g. $|z_1 + z_2| > |z_1| - |z_2|$ eg $z_1 = 1$, $z_2 = 1$

C false if signs different e.g. $z_1 = 1$ $z_2 = -1$

$|z_1 - z_2| > ||z_1| - |z_2||$

(D) correct answer This is in fact. The reverse triangle inequality.

5) $\int_0^{\pi/2} f(x)\cos x\,dx = 2$ $\quad f(\pi/2)=3$ $\quad f(0)=-1$ $\quad$ let $u=f(x)$ $\quad v'=\cos x$ $\quad \frac{du}{dx}=f'(x)$ $\quad v=\sin x$

$$uv\Big|_0^{\pi/2} - \int_0^{\pi/2} v\frac{du}{dx}dx = 2$$

$$\Big[f(x)\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} f'(x)\sin x\,dx = 2$$

$$f(\pi/2)\times\sin\pi/2 - f(0)\times\sin 0 - I = 2$$

$$3\times 1 - -1\times 0 - 2 = I.$$

$\therefore I = 3-2 = 1$ $\quad\therefore$ (A)

6) (D) as in A $\sin^2 t$ can be expressed in terms of $\cos 2t$, in B and C can use auxiliary angle to express as one trig function (C has an offset). However in D periods different so although periodic, it is not SHM.

7) need 5 correct unique vectors to go from D to K

eliminate B as only 4

we note $\vec{AB}$ wrong direction need $\vec{BA}$ or $-\vec{AB}$ so we can eliminate A and D $\therefore$ (C)

checking $\vec{AE}=\vec{DG}$ $\quad -\vec{FC}=\vec{GF}$ $\quad -\vec{AB}=\vec{FE}$ $\quad \vec{IL}=\vec{FH}$ and $\vec{HK}$ ✓

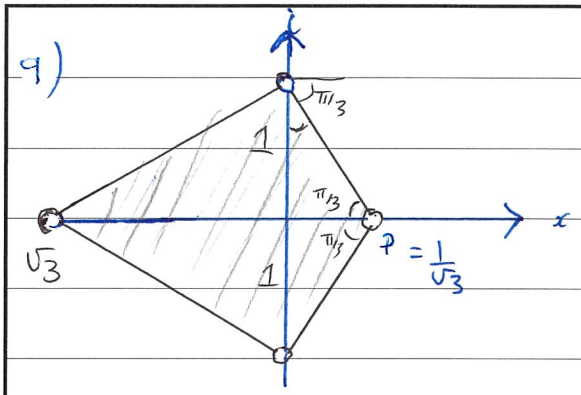
8) arccos has range $[0,\pi]$ $\therefore$ eliminate C and D.

$\begin{pmatrix}0\\-1\\0\end{pmatrix}$ on curve so $x=0$ and $z=0$ at same time.

$\sin t = 0$ when $t=0$ or 1 $\quad \arcsin(\sin\pi t)$ also $=0$ when $t=0,\pi$

$\therefore$ (B)

9)



$\tan \pi/3 = \frac{1}{p}$

$\sqrt{3} = \frac{1}{p} \quad \therefore p = \frac{1}{\sqrt{3}}$

$A = \frac{1}{2}(a+b) = \frac{1}{2} \times (1+1) \times \left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)$

$= \frac{\sqrt{3}\times\sqrt{3}+1}{\sqrt{3}}$

$= \frac{4}{\sqrt{3}} \quad \therefore$ (B)

note: after area, the holes at intersect. do not matter.

10) $f(2x) = 3f(x)$

$\therefore f(x) = \frac{f(2x)}{3}$

$\int_0^1 f(x)\,dx = 2$ (given)

$\int_0^1 \frac{f(2x)}{3}\,dx = 2$

Let $u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x = 0 \quad u = 0$

$x = 1 \quad u = 2$

$\int_{u=0}^{u=2} \frac{f(u)}{3} \times \frac{du}{2} = 2$

$\int_0^2 f(u)\,du = 2 \times 6$ but u a dummy variable. $\therefore \int_0^2 f(x)\,dx = 12$

and $\int_1^2 f(x)\,dx = \int_0^2 f(x)\,dx - \int_0^1 f(x)\,dx$

$= 12 - 2 = 10 \quad \therefore$ (C)

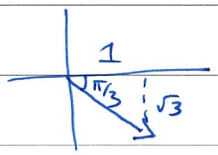
— end MC —

Q11) (a) $\int 2\cos 5x \cos 4x\,dx = 2\cdot\frac{1}{2}\int(\cos(5x+4x) - \sin(5x-4x))\,dx$

$= \int \cos 9x\,dx - \int \sin x\,dx$

$= \frac{\sin 9x}{9} + \cos x + C$

(b) (i) $1-\sqrt{3}i = 2e^{-i\pi/3}$



(ii) $(1-\sqrt{3}i)^6 = (2e^{-i\pi/3})^6$

$= 2^6 \cdot e^{-6\pi/3 i} = 64e^{-2\pi i} = 64e^{2\pi i}$

$= 64$

(c) to be colinear all on same line.

so for example $(-3, \alpha, \beta)$ on line through $(5,4,2)$ and $(3,2,2)$

i.e $\begin{pmatrix}-3\\ \alpha\\ \beta\end{pmatrix} = \begin{pmatrix}5\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}5-3\\4-2\\2-2\end{pmatrix}$

$\begin{pmatrix}-3\\ \alpha\\ \beta\end{pmatrix} = \begin{pmatrix}5\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-8\\0\end{pmatrix}$

so $\beta = 2$ from z and $2\lambda + 5 = -3$ from x $\therefore \lambda = -4$

from y. $\alpha = 4 + -8\times\lambda = 4 + 32 = 36$.

$\therefore \alpha = 36$ and $\beta = 2$

Q11) d)

$z^2 - 6z + 13 = 0$

$a=1 \quad b=-6 \quad c=13$.

$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 1\times 13}}{2} = \frac{6 \pm \sqrt{36-52}}{2}$

$= \frac{6 \pm \sqrt{-16}}{2}$

$z = 3+2i$ or $3-2i$

(e) $\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}| \cos\theta$

$\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \sqrt{1^2+1^2+(-4)^2} \cdot \sqrt{5^2+4^2+3^2} \cos\theta$

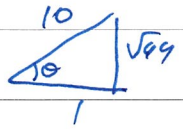
$\sqrt{18} \cdot \sqrt{50} \cos\theta = 5+4-12$

$\cos\theta = \frac{-3}{\sqrt{900}} = -\frac{3}{30}$

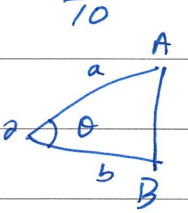
$\cos\theta = -\frac{1}{10}$ $\therefore \theta = 96^\circ$

So $|\vec{OA}||\vec{OB}| = 30$

(ii) in Q2 so $\sin\theta$ +ve.



$\therefore \sin\theta = \frac{\sqrt{99}}{10} = \frac{3\sqrt{11}}{10}$



$A = \frac{1}{2} ab \sin\theta = \frac{1}{2} \cdot 30 \cdot \frac{3\sqrt{11}}{10}$

$A = \frac{9\sqrt{11}}{2} u^2$

(END Q11)

Question 12

(a) $\int_1^e \frac{(\ln x)^3}{x}\,dx = \int_0^1 u^3\,du$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$x = 1 \quad u = 0$

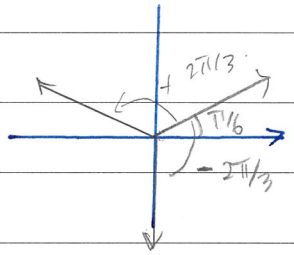
$x = e \quad u = 1$

$= \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}$

(b) $i = \text{cis}(\pi/2)$

$z^3 = \text{cis}(\pi/2)$

$\therefore z = \text{cis}(\pi/2 \div 3)$

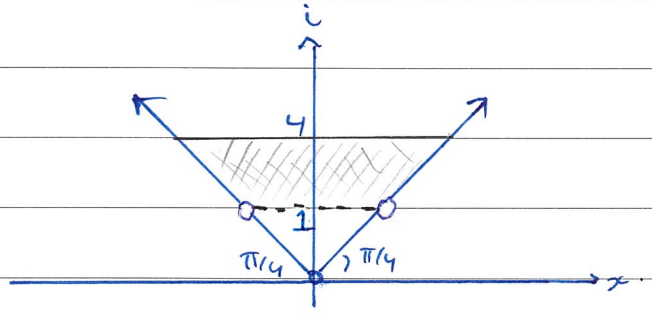


$z = \text{cis}(\pi/6),\ \text{cis}(\pi/6 + 2\pi/3),\ \text{cis}(\pi/6 - 2\pi/3)$

$= \text{cis}\,\pi/6,\ \text{cis}\,5\pi/6,\ \text{cis}(-\pi/2)$

$= \sqrt{3} + \frac{1}{\sqrt{3}}i,\ -\sqrt{3} + \frac{1}{\sqrt{3}}i,\ -i$

(c)



(d) $\ddot{x} = -9(x-2)$

$= -3^2(x-2)$ SHM $n = 3$

$c = 2$

(Q12d) cont....

So

$x = a\sin(nt+\alpha) + c$

$x = a\sin(3t+\alpha) + 2$

$\dot{x} = 3a\cos(3t+\alpha)$

Note we are not told initially, and are only after amplitude, not interested in position at certain time, so the phase and α not needed.

let $B = 3t+\alpha$
(we don't need α, clean)

$x = a\sin(B) + 2$

$\dot{x} = 3a\cos(B)$ at the same time or B $x = -1$ $\ddot{x} = 6\sqrt{3}$

So $-1 = a\sin(B) + 2 \Rightarrow a\sin(B) = -3$

$6\sqrt{3} = 3a\cos(B)$ $a\cos(B) = 2\sqrt{3}$

↖ pick +ve as we square anyway and only after amplitude.

$(a\sin(B))^2 + (a\cos(B))^2 = (-3)^2 + (2\sqrt{3})^2$

$a^2(\sin^2 B + \cos^2 B) = 21$

$\therefore a = \sqrt{21}$ m

for full Period $\xrightarrow{a}\ |\ \xrightarrow{a}$ $\quad \xleftarrow{a}\ \xleftarrow{a}$ distance travelled $= 4a$

$\therefore$ distance travelled in full period $= \underline{4\sqrt{21}}$

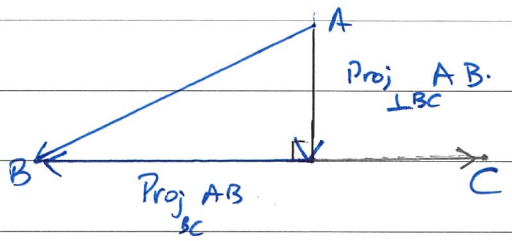
e) (i) $\vec{OA} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$ $\vec{OB} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\vec{OC} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\text{Proj}_{\vec{BC}}\vec{AB} = \dfrac{\vec{AB}\cdot\vec{BC}}{|\vec{BC}|^2} \times \vec{BC} = \dfrac{1\times0 + 1\times-1 + 0}{(\sqrt{2})^2}\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\1/2\\-1/2\end{pmatrix}$

Q12(e) cont

(ii)



The shortest line ~~from~~ A to line BC is
The magnitude of Projection of AB onto the Perpendicular line.
of BC ($\underset{\perp BC}{\text{Proj}}\ AB$)

$$\underset{\perp BC}{\text{Proj}}\ AB = AB - \underset{BC}{\text{Proj}}\ AB$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$d_{min} = \left| \underset{\perp BC}{\text{Proj}}\ AB \right| = \sqrt{1^2 + 1/2^2 + 1/2^2} = \sqrt{\frac{6}{4}}$$

$$\underline{d_{min} = \frac{\sqrt{6}}{2}}$$

Question 13

(a) $\sum_{i=1}^{2026} i^n = i + i^2 + i^3 + i^4 + i^5 + \ldots\ldots + i^{2024} + i^{2025} + i^{2026}$

$= \underbrace{i - 1 - i + 1}_{=0} + \underbrace{i + \ldots\ldots + 1}_{=0 \text{ as divisible by } 4} + i - 1$

$= i - 1$

Q13 continued

(b) (i) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$

$(\sqrt{x})^2 + \left(\frac{-1}{\sqrt{x}}\right)^2 - \frac{2\sqrt{x}}{\sqrt{x}} \geq 0$

$x + \frac{1}{x} - 2 \geq 0$

$\therefore x + \frac{1}{x} \geq 0$ as required.

(ii) LHS $= \left(\frac{a}{b} - \frac{c}{d}\right)\left(\frac{b}{a} - \frac{d}{c}\right)$

$= \frac{a}{b}\frac{b}{a} - \frac{c}{d}\cdot\frac{b}{a} - \frac{a}{b}\cdot\frac{d}{c} + \frac{c}{d}\cdot\frac{d}{c}$

$= 1 - \frac{bc}{da} - \frac{ad}{bc} + 1$

$= 2 - \left(\frac{bc}{ad} + \frac{ad}{bc}\right)$

from Part (i) let $x = \frac{bc}{ad}$

$x + \frac{1}{x} \geq 2$

$-\left(x + \frac{1}{x}\right) \leq -2$

$\leq 2 - 2$

≤ 0

$\therefore \left(\frac{a}{b} - \frac{c}{d}\right)\left(\frac{b}{a} - \frac{d}{c}\right) \leq 0$ as required.

(Q13) b(ii) alternate solution

$$LHS = \left(\frac{a}{b} - \frac{c}{d}\right)\left(\frac{b}{a} - \frac{d}{c}\right)$$

$$= \left(\frac{ad - bc}{bd}\right)\left(\frac{bc - ad}{ac}\right)$$

$$= -\left(\frac{ad - bc}{bd}\right)\left(\frac{ad - bc}{ac}\right)$$

$$= -\frac{(ab - bc)^2}{abcd}$$

as $(ab-bc)^2 \geq 0$ (perfect square)

≤ 0 as required. $abcd > 0$ (given)

(c)

$$\int_1^4 \frac{3}{x + 4\sqrt{5-x}}\, dx$$

$u^2 = 5 - x$

$x = 5 - u^2$

$dx = -2u\, du$

$x = 1 \quad u^2 = 4 \therefore u = 2 \ (u > 0)$

$x = 4 \quad u^2 = 1 \therefore u = 1$

$$= \int_{u=2}^{u=1} \frac{3}{(5-u^2) + 4\sqrt{u^2}} \cdot -2u\, du$$

$$= \int_1^2 \frac{6u}{5 - u^2 + 4u}\, du$$

$$= \int_1^2 \frac{6u}{(5-u)(u+1)}\,du$$

$$\frac{6u}{(5-u)(u+1)} = \frac{A}{5-u} + \frac{B}{u+1}$$

$$6u = (u+1)A + (5-u)B$$

let $u=5$ $\quad 6A = 30 \therefore A = 5$

let $u=1$ $\quad 6B = -6 \therefore B = -1$

$$= \int_1^2 \left(\frac{5}{5-u} - \frac{1}{u+1}\right) du$$

$$= \left[-5\ln(5-u) - \ln(u+1)\right]_1^2$$

$$= \left[5\ln(5-u) + \ln(u+1)\right]_2^1$$

$$= 5\ln 4 + \ln 2 - (5\ln 3 + \ln 3)$$

$$= \ln\left(\frac{4^5 \times 2}{3^5 \times 3}\right)$$

$$= \ln\left(\frac{2048}{729}\right) \text{ as required.}$$

(d) $z = \cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

$= \text{cis}(-\theta)$

$z^n + \frac{1}{z^n} = z^n + z^{-n}$

$= \text{cis}(-n\theta) + \text{cis}(n\theta)$ (de Moivre's)

Q13 d (ii) continued

$$z^n + \frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta) + \cos(n\theta) + i\sin(n\theta)$$

$$= \cos n\theta - i\sin n\theta + \cos n\theta + i\sin n\theta \quad (\text{sin odd, cos even})$$

$$= 2\cos n\theta \text{ as required.}$$

(ii) $$\left(z + \frac{1}{z}\right)^5 = z^5 + 5z^4 \times \frac{1}{z} + 10z^3 \times \frac{1}{z^2} + 10z^2 \times \frac{1}{z^3} + 5z \times \frac{1}{z^4} \times \frac{1}{z^5}$$

$$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$\therefore A = 2 \quad B = 10 \text{ and } C = 20$$

(iii) $$\left(z^5 + \frac{1}{z^5}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$$

so using Part (ii)

$$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$\text{so } \cos^5\theta = \frac{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}{16}$$

$$\int_0^{\pi/2} \cos^5\theta \, d\theta = \frac{1}{16} \int_0^{\pi/2} \cos 5\theta + 5\cos 3\theta + 10\cos\theta \, d\theta$$

$$= \frac{1}{16}\left[\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin\theta\right]_0^{\pi/2}$$

$$= \frac{1}{16}\left[\left(\frac{\sin 5\pi/2}{5} + \frac{5\sin 3\pi/2}{3} + 10\sin\pi/2\right) - (0)\right]$$

$$= \frac{1}{16}\left[\frac{1}{5} - \frac{5}{3} + 10\right]$$

$$= \frac{8}{15}$$

Question 14

(a)(i) $a = v\sqrt{4-v^2}$.

(ii) $\ddot{x}$ always +ve

as $\dot{x}$ initially > 0 so $v\sqrt{4-v^2} > 0$

as $t \to \infty$ $v \to 2$ $x \to \infty$

The particles velocity increases and approaches 2 m/s

$$\frac{dv}{dx} = \sqrt{4-v^2}$$

$$\int_1^v \frac{dv}{\sqrt{4-v^2}} = \int_0^{\pi/6} x\,dx$$

$$\left[\arcsin\left(\frac{v}{2}\right)\right]_1^v = \left[x\right]_0^{\pi/6}.$$

$$\arcsin\left(\frac{v}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \pi/6.$$

$$\arcsin\left(\frac{v}{2}\right) = \pi/3$$

$$\frac{v}{2} = \frac{\sqrt{3}}{2}$$

$$\therefore v = \sqrt{3}\ \text{m/s}$$

Q14 continued:

(b)

$$V = \pi \int_0^{\sqrt{3}} \left(\sqrt{2x \arctan x}\right)^2 dx$$

$$= \pi \int_0^{\sqrt{3}} 2x \arctan x \, dx$$

$u = \arctan x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = 2x$

$v = x^2$

$$= \pi \left[[uv]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{du}{dx} v \, dx \right]$$

$$= \pi \left[\left[x^2 \arctan x \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} dx \right]$$

$$= \pi \left[\left[(\sqrt{3})^2 \cdot \pi/3 - 0 \right] - \int_0^{\sqrt{3}} \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx \right]$$

$$= \pi \left[\pi - \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+x^2}\right) dx \right]$$

$$= \pi^2 - \pi \left[x - \arctan x \right]_0^{\sqrt{3}}$$

$$= \pi^2 - \pi \left[\left(\sqrt{3} - \pi/3\right) - (0-0) \right]$$

$$= \pi^2 - \sqrt{3}\pi + \pi^2/3$$

$$= \frac{4\pi^2}{3} - \sqrt{3}\pi \quad \text{u}^3$$

Q14 (c)

assume $4x^2 - y^2 = 1$ $\quad x, y \in \mathbb{Z}$

$(2x)^2 - y^2 = 1$

$(2x+y)(2x-y) = 1$

as $x, y \in \mathbb{Z}$ $\quad (2x+y) \in \mathbb{Z}$ and $(2x-y) \in \mathbb{Z}$

$\therefore$ only possible solutions are.

Case 1

$(2x+y) = 1$ — ①

$(2x-y) = 1$ — ②.

or

Case 2

$(2x+y) = -1$ — ③

$(2x-y) = -1$ — ④

① + ②

$4x = 2 \quad \therefore x = 1/2$

which contradicts assumption.

③ + ④

$4x = -2$

$\therefore x = -1/2$

which also contradicts assumption.

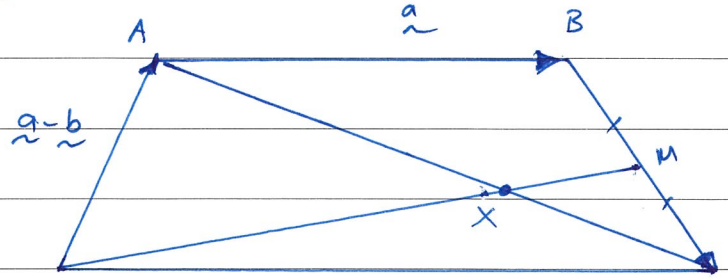
since both cases lead to non-integer values of x

original assumption must be wrong.

Therefore by contradiction $4x^2 - y^2 = 1$ has no integer solutions.

Q14 continued.

(d)



$OX:OM = n:1$

$\underset{\sim}{b} = \vec{BC}$ $\quad \underset{\sim}{a} = \vec{AB}$ $\quad \underset{\sim}{a} - \underset{\sim}{b} = \vec{OA}$

(i) $\vec{OC} = \vec{OA} + \vec{OB} + \vec{BC}$

$= \underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{a} + \underset{\sim}{b}$

$= 2\underset{\sim}{a}$

(ii) $\vec{OX} = \frac{n}{n+1}\vec{OM}$

$= \frac{n}{n+1}(\vec{OB} + \vec{BM})$

$= \frac{n}{n+1}(\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b})$

$= \frac{n}{n+1}(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b})$

$\vec{OX} = \frac{2n}{n+1}\underset{\sim}{a} - \frac{n}{2(n+1)}\underset{\sim}{b}$

$\vec{AX} = \vec{AO} + \vec{OX}$

$= \underset{\sim}{b} - \underset{\sim}{a} + \frac{2n}{n+1}\underset{\sim}{a} - \frac{n}{2(n+1)}\underset{\sim}{b}$

$= \frac{2n-(n+1)}{n+1}\underset{\sim}{a} + \frac{2(n+1)-n}{2(n+1)}\underset{\sim}{b}$

$\vec{AX} = \frac{n-1}{n+1}\underset{\sim}{a} + \frac{n+2}{2(n+1)}\underset{\sim}{b}$ as required.

(iii) $\vec{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

X also lies on line $\vec{AC}$

$\vec{AX} = \lambda \vec{AC}$

$\vec{AX} = \lambda(\underset{\sim}{a} + \underset{\sim}{b})$

equating coefficients for $\underset{\sim}{a}$ and $\underset{\sim}{b}$ components using Part (i)

$\left(\frac{n-1}{n+1}\right) = \lambda$ and $\frac{n+2}{2(n+1)} = \lambda$.

i.e

$\frac{(n-1)}{n+1} = \frac{n+2}{2(n+1)}$ $(n>0)$

$2(n-1) = n+2$

$2n-2 = n+2$

$\therefore n = 4$

Question 15

(a) $Q \Rightarrow P$ Contrapositive $\neg P \Rightarrow \neg Q$

$Q = \neg(A \Rightarrow C)$

$P = (\neg(A \Rightarrow B) \text{ OR } \neg(B \Rightarrow C))$

$\neg Q = \neg\neg(A \Rightarrow C) = A \Rightarrow C$

$\neg P = \neg(\neg(A \Rightarrow B) \text{ OR } \neg(B \Rightarrow C))$

$= ((A \Rightarrow B) \text{ AND } (B \Rightarrow C))$

∴ Contraposite is

$(A \Rightarrow B) \text{ AND } (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

also accept

$\neg(\neg(A \Rightarrow B) \text{ OR } \neg(B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$

(b) (i)

$$I_n = \int_0^1 x(1-x)^n \, dx$$

$\frac{dv}{dx} = (1-x)^n$

$v = -\frac{(1-x)^{n+1}}{n+1}$

$u = x$

$\frac{du}{dx} = 1$

$$= \left[uv\right]_0^1 - \int_0^1 v \frac{du}{dx} \, dx.$$

$$= \left[\frac{-(1-x)^{n+1}}{n+1} \cdot x\right]_0^1 - \int_0^1 -\frac{(1-x)^{n+1}}{n+1} \, dx.$$

$$= [0 - 0] + \int \frac{(1-x)^{n+1}}{n+1}\,dx$$

$$= \left[\frac{-(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1 = \left[\frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_1^0$$

$$= \frac{(1-0)^{n+2} - 0}{(n+1)(n+2)}$$

$$= \frac{1}{(n+2)(n+1)} \quad \text{as required.}$$

(ii) $$S_n = \sum_{k=1}^{n} \frac{1}{(n+1)(n+2)}$$

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = (n+2)A + (n+1)B.$$

let $n = -1$

$A = 1$

let $n = -2$

$-B = 1 \therefore B = -1.$

$$= \sum_{k=1}^{n} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+2}$$

$$= \frac{n+2-2}{2(n+2)}$$

$$S_n = \frac{n}{2(n+2)}$$

(Q15) continued.

(c)

RTP $\tan\theta + 2\tan 2\theta + \cdots + 2^{n-1}\tan(2^{n-1}\theta) = \cot\theta - 2^n\cot(2^n\theta)$

for $n=1$

LHS $= \tan\theta$

RHS $= \cot\theta - 2\cot(2\theta)$

$= \cot\theta - 2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)$

$= \cot\theta - \frac{1}{\tan\theta} + \frac{\tan^2\theta}{\tan\theta}$

$= \cot\theta - \tan\theta + \tan\theta$

$= \tan\theta$

$= $ RHS

$\therefore$ true for $n=1$

assume true for $n=k$.

i.e $\tan\theta + 2\tan 2\theta + \cdots + 2^{k-1}\tan(2^{k-1}\theta) = \cot\theta - 2^k\cot(2^k\theta)$

RTP for $n=k+1$

i.e RTP

$\tan\theta + 2\tan\theta + \cdots + 2^{k-1}\tan(2^{k-1}\theta) + 2^{k+1-1}\tan(2^{k+1-1}\theta)$

$= \cot\theta - 2^{k+1}\cot(2^{k+1}\theta)$

for $n = k+1$

LHS $= \tan\theta + 2\tan 2\theta + \cdots 2^{k-1}\tan(2^{k-1}\theta) + 2^k(2^k\theta)$

$= \cot\theta - 2^k\cot(2^k\theta) + 2^k\tan(2^k\theta)$ from assumption

$$LHS = \cot\theta - 2^k\left(\cot(2^k\theta) - \tan(2^k\theta)\right)$$

$$= \cot\theta - 2^k\left(\frac{\cos(2^k\theta)}{\sin(2^k\theta)} - \frac{\sin(2^k\theta)}{\cos(2^k\theta)}\right)$$

$$= \cot\theta - \frac{2^{k+1}}{2}\left(\frac{\cos(2^k\theta)\cdot\cos(2^k\theta) - \sin(2^k\theta)\cdot\sin(2^k\theta)}{\sin(2^k\theta)\cos(2^k\theta)}\right)$$

$$= \cot\theta - 2^{k+1}\left(\frac{\cos(2\cdot 2^k\theta)}{2\sin(2^k\theta)\cos(2^k\theta)}\right)$$

$$= \cot\theta - 2^{k+1}\left(\frac{\cos(2^{k+1}\theta)}{\sin(2^{k+1}\theta)}\right)$$

$$= \cot\theta - 2^{k+1}\cot(2^{k+1}\theta)$$

$$= RHS$$

$\therefore$ true for $n=1$ if true for $n=k$.

but true for $n=1$ so true for $n=2, 3, \ldots$

$\therefore$ by the process of mathematical induction.

true $\forall n \in \mathbb{Z}^+$

alternate inductive step.

Note after assumption we can use the result from step 1 but we need to structure carefully.

$$LHS = \cot\theta - 2^k\cot(2^k\theta) + 2^k\tan(2^k\theta)$$

$$= \cot\theta - 2^k\left(\cot(2^k\theta) - \tan(2^k\theta)\right)$$

now let $\alpha = 2^k\theta$

From $n=1$ step.
$\tan\theta = \cot\theta - 2\cot 2\theta$
$\therefore \cot\alpha - \tan\alpha = 2\cot(2\alpha)$

$LHS = \cot\theta - 2^k(\cot\alpha - \tan\alpha)$

$= \cot\theta - 2^k(2\cot 2\alpha)$

$= \cot\theta - 2.2^k \cot(2.2^k\theta)$ (remember $\alpha = 2^k\theta$)

$= \cot\theta - 2^{k+1}\cot 2^{k+1}\theta$

$= RHS$

etc

(d) Fundamental Theorem of Calculus $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

let $g(x) = \int_1^x \ln(t)\,dt$ $\quad f(t) = \ln(t)$

$$\frac{d}{dx}\int_1^{x^2} \ln t\,dt = \frac{d}{dx}g(x^2)$$

$= g'(x^2)\cdot\frac{d}{dx}x^2$ (chain rule).

$= 2x\,g'(x)^2$

$= 2x\,f(x^2)$ (fundamental theorem of calculus)

$= 2x\ln(x^2)$

$= 2.2x\ln(x)$

$= 4x\ln x$ as required.

P23

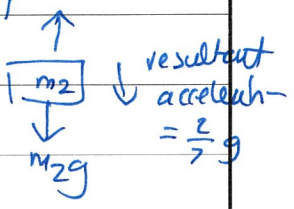
(Q15)(e) $F_k \leftarrow [m_1]—[m_2] \rightarrow m_2 g$ → direction of travel

$\Sigma F = m_2 g - F_k$

$= 6g - \frac{1}{4} \times 8 \times g = 4g$

$F = ma$ where M is the mass of whole system $(M = 6+8 = 14)$

$\therefore 4g = 14a$ i.e $a = \frac{4g}{14}$ $a = \frac{2}{7}g$



For tension look at Part of system

$\Sigma F = (mg - T) = 6g - T$ — (1)

also $F = ma$.

$F = 6 \times \frac{2}{7}g = \frac{12g}{7}$ — (2)

(1) = (2)

$6g - T = \frac{12g}{7}$, so $T = 6g - \frac{12}{7}g$

$T = \frac{42 - 12}{7} g$ N

$T = \frac{30}{7} g$ N

Question (16)

(a) (i) $2n$ and $2n+2$ $n \in \mathbb{Z}^+$ are 2 positive consecutive integers.

$2n(2n+1) = 4 \cdot n \cdot (n+1)$ n and $n+1$ are consecutive integers

$= 4 \cdot 2k$ $k \in \mathbb{Z}^+$ if n even then $n+1$ odd if n odd then $n+1$ even.

$= 8k$ $\therefore n(n+1) = 2k$ $k \in \mathbb{Z}^+$

which is divisible by 8

(Q16)(ii) $(p^2-1) = (p+1)(p-1)$

p is Prime and > 3 $\therefore$ p is odd

$\therefore$ $(p+1)$ and $(p-1)$ are consecutive even integers.

$\therefore$ from Part(i) $(p+1)(p-1) = 8m$ $\quad m \in \mathbb{Z}^+$

now consider the 3 consecutive integers $(p-1), p, (p+1)$

one of the 3 must be divisible by 3

however p is Prime > 3 $\therefore$ not divisible by 3

i.e either $(p-1)$ divisible by 3 OR $(p+1)$ divisible by 3

$\therefore$ $(p-1)(p+1) = 3p$ $\quad p \in \mathbb{Z}^+$.

as 8 and 3 are co prime.

$(p-1)(p+1) = 3 \times 8 \times s$ $\quad s \in \mathbb{Z}^+$.

$\therefore$ $p^2 - 1 = 24s$

i.e p^2-1 is divisible by 24 if p is a Prime > 3

(b)(i) $\uparrow$ +ve $\quad \uparrow F_r \quad \downarrow F_{gravity}$.

$\Sigma F = F_r + F_{gravity}$

$= \underbrace{\frac{mg}{x^3}(1+v^2) - mgv^2}_{F_r} - \underbrace{mg}_{F_g}$

$F_{total} = mg\left(\frac{1+v^2}{x^3} - 1 - v^2\right) =$

$ma = mg\left(\frac{1+v^2}{x^2} - (1+v^2)\right)$

$v\frac{dv}{dx} = g(1+v^2)\left(\frac{1}{x^3} - 1\right)$

$\frac{v}{1+v^2}\frac{dv}{dx} = -g\left(1 - \frac{1}{x^3}\right)$ as required.

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Q16 (b) (ii)

$$\frac{V}{1+V^2}\frac{dV}{dx} = -g\left(1-\frac{1}{x^3}\right)$$

initially $V = U$

$V_{\text{max height}} = 0$

initially $x = 1$

$x_{\text{max height}} = H$

$$\int_U^0 \frac{V}{1+V^2}\,dV = -\int_1^H g\left(1-\frac{1}{x^3}\right)dx$$

U is positive so max height when V=0

what happens after this is beyond the scope of the question

$$\frac{1}{2}\int_U^0 \frac{2V}{1+V^2}\,dV = \int_1^H \frac{g}{x^3} - g\,dx$$

$$\left[\frac{1}{2}\ln(1+V^2)\right]_U^0 = \int_1^H gx^{-3} - g\,dx$$

$$\frac{1}{2}\ln(1) - \frac{1}{2}\ln(1+U^2) = \left[\frac{-gx^{-2}}{2} - gx\right]_1^H$$

$$-\frac{1}{2}\ln(1+U^2) = \left[-\frac{gx^{-2}}{2} - gx\right]_1^H$$

$$\ln(1+U^2) = \left[gx^{-2} + 2gx\right]_1^H$$

$$= \left[\frac{g}{H^2} - g + 2gH - 2g\right]$$

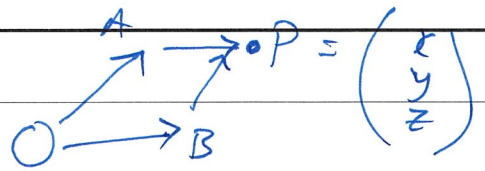
$$= 2g(H-1) + g\left(\frac{1}{H^2} - 1\right)$$

$$\ln(1+U^2) = 2g(H-1) + g\left(\frac{1-H^2}{H^2}\right)$$

$$\ln(1+U^2) = 2g(H-1) - g\left(\frac{H^2-1}{H^2}\right)$$

as required.

Q16(c) $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix}$



$\overrightarrow{AP} = \begin{pmatrix} x-6 \\ y-4 \\ z+2 \end{pmatrix}$ $\overrightarrow{BP} = \begin{pmatrix} x-4 \\ y-8 \\ z-2 \end{pmatrix}$

$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$

$(x-6)(x-4) + (y-4)(y-8) + (z+2)(z-2) = 0$

$x^2 - 10x + 24 + y^2 - 12y + 32 + z^2 - 4 = 0$.

$x^2 - 10x + y^2 - 12y + z^2 + 52 = 0$.

$(x-5)^2 - 25 + (y-6)^2 - 36 + z^2 + 52 = 0$.

$(x-5)^2 + (y-6)^2 + z^2$

This is a circle radius 3 centred on $\begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}$

$\therefore \quad |\underset{\sim}{r} - \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}| = 3$ is the vector equation of S_1

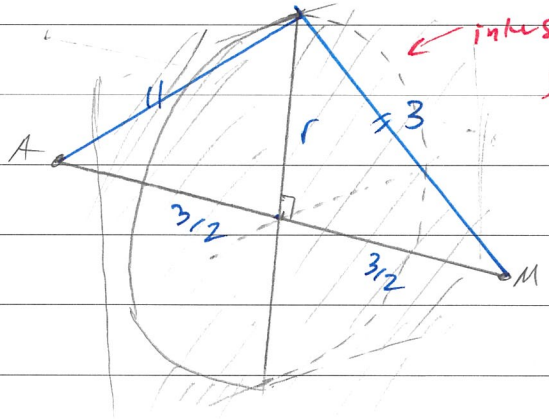
alternately: we observe that in 2D the angle subtended by 2 points on a diameter of a circle equals 90° (Thales theorem)
∴ in 3D this will be a sphere, centre is midpoint of AB and radius equals $\frac{1}{2}$ length of segment AB

$\overrightarrow{BA} = \begin{pmatrix} 6-4 \\ 4-8 \\ -2-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$ $\frac{|\overrightarrow{BA}|}{2} = \frac{\sqrt{2^2 + (-4)^2 + (-4)^2}}{2} = \frac{\sqrt{36}}{2} = \underline{\underline{3}}$

and $M = \begin{pmatrix} \frac{6+4}{2} \\ \frac{4+8}{2} \\ \frac{-2+2}{2} \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}$ $\therefore S_1 \in |\underset{\sim}{r} - \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}| = 3$

Q16 (c) (ii) $|\vec{AN}| = |\vec{MN}|$ in 2D N would be the line.

That was the perpendicular bisector of AM in 3D this extends up and down to a plane whose intersection with sphere from part (i) gives us our circle.



← intersection with sphere.

$r^2 = 3^2 - \left(\frac{3}{2}\right)^2$

$= 9 - \frac{9}{4} = \frac{27}{4}$.

$r = \frac{\sqrt{27}}{4}$

$r = \frac{3\sqrt{3}}{2}$

(d) $\int_a^b f(x)\,dx$

let $x = a+b-u$

$dx = -du$

$= \int_{u=b}^{a} f(a+b-u) \cdot -du$

$x = b \rightarrow u = a$

$x = a \rightarrow u = b$.

$= -\int_b^a f(a+b-u)\,du$

$= \int_a^b f(a+b-u)\,du$

$= \int_a^b f(a+b-x)\,dx$

as definite independent of the variable used "dummy variable"

Q16 (d)

(ii) $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\,dx$ [alternately split into $\sqrt{\frac{\sin x}{\cos x}}$ and $\times \frac{\sqrt{\cos x}}{\sqrt{\cos x}}$]

$= \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan(\pi/2 - x)}}\,dx$ from part (i) $\left(\frac{a+b-x}{\pi/3+\pi/6=\pi/2}\right)$

$= \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$

$= \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{\sqrt{1}}{\sqrt{\tan x}}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}\,dx$ $\left(\times \frac{\sqrt{\tan x}}{\sqrt{\tan x}}\right)$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}+1-1}{\sqrt{\tan x}+1}\,dx$

$I = \int_{\pi/6}^{\pi/3} 1 - \frac{1}{\sqrt{\tan x}+1}\,dx = \int_{\pi/6}^{\pi/3} 1\,dx - I$

$2I = \int_{\pi/6}^{\pi/3} 1\,dx = \pi/3 - \pi/6$

$\therefore I = \pi/12$

The end